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Research Article Extension Of Semi-Continuous Functions

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Abstract: An attempt is made to analyse the extension of semi-continuous function defined on a closed set.

Key words: closed set, semi-continuous function, lower semi-continuous, upper semi-continuous function.

A function $f : X \to \mathbb{R}$ to be lower semi-continuous if for each real b, $\{x / f(X) \leq b\}$ is closed. Similarly for $f : X \to \mathbb{R}$ is upper semi-continuous if $\{x / f(X) \geq b\}$ is closed for every real b. In this article, an attempt is made to analyse the extensions for semi-continuous functions defined on a closed set.
Let X be an uncountable set endowed with the co-countable topology. Let A be an open subset of X. Then A = Φ or A = X or A = B^C for some countable subset B of X.

Lemma 1
Let X be a uncountable set endowed with co-countable topology. If f is ***lsc*** on X, then f(X) has maximum.

Proof
Suppose f is not bounded above.
Then $f^{-1}(n, \infty) \neq \Phi$ for all $n \in \mathbb{N}$, which implies that , $f^{-1}(-\infty, n]) \neq \Phi$ for all $n \in \mathbb{N}$.

Since f is ***lsc***, $f^{-1}(n, \infty)$ is open for all $n \in \mathbb{N}$,
$\Rightarrow f^{-1}(-\infty, n])$ is closed for all $n \in \mathbb{N}$.

Since X is the only uncountable closed set, $f^{-1}(-\infty,n])$ is countable for all $n \in \mathbb{N}$.
Clearly $X = f^{-1}(-\infty\ \infty) = \bigcup_{n \in \mathbb{N}} f^{-1}(-\infty, n])$.

Since countable union of countable set is countable set. X is countable, which is a contradiction to the assumption. Therefore f is a bounded above.
Hence, f(X) has maximum.

Example 1
The function f given by $f(n) = n$, $n \in \mathbb{N}$ on the closed subset $\mathbb{N}$ of $\mathbb{R}$ endowed with the co-countable topology, is ***lsc*** and it has no ***lsc*** extension.

Proof
In co-countable topology, open sets are Φ and all complements of countable sets.
Now, $\mathbb{N}$ is countable.
Therefore $\mathbb{R} - \mathbb{N}$ is open.
Clearly, the function $f = \mathbb{N} \to \mathbb{R}$ given by $f(n) = n$, $n \in \mathbb{N}$ is ***lsc***.
Suppose $F : \mathbb{R} \to \mathbb{R}$.
Clearly F is not ***lsc***, because we cannot find out countable closed subset in $\mathbb{R}$.
Remarks
Hereafter, A stands for the closed subset of X.
Theorem 1
Let F be an ***lsc*** function on A and suppose it is upper bounded on the boundary of A. Then, f has an ***lsc*** extension F to X. which is upper bounded on X – A, such that $F(X) \subset f(A)$.

Proof
Let $f = A \to \mathbb{R}$ be the given ***lsc*** function such that f is upper bounded on the boundary of A.
Let M = Sup {f (b(a))}

Define the function $F = X \to \mathbb{R}$ as follows:

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$F(X) = \begin{cases} f(x)\ if\ x \in A, \\ M\ if\ x \notin x - A \end{cases}$

We will to prove that F is a ***lsc*** function on X.
Let a∈ℝ.
Case (i) Suppose a < M,

Then $F^{-1}((-\infty, a]) = f^{-1}((-\infty, a])$ is closed in A, A is closed in X.

Therefore F is ***lsc*** on X.
Case (ii) Suppose a ≥ M,
Then $F^{-1}((-\infty, a]) = f^{-1}((-\infty, a])$ U(X – A) is closed.

Since the boundary b(X – A) of X – A is b(A) which is contained in $f^{-1}((-\infty, a])$, which implies that F is ***lsc*** on X.
Clearly, since A is closed, b(A) ⊂ A.
Therefore M ∈cl (f(A)), which implies that f(X) C d(f(A)).
Also F: X f→ℝ is upper bounded on X – A.
The converse of the Theorem 1 is false.
Example 2
The ***lsc*** function F on ℝ given by F(X) = n if X ∈ (n, n+1), n ∈ℤ, is an extension of the ***lsc*** function f defined on the set of integers ℤ given by f(n) = n, n ∈ℤ. The boundary b(ℤ) of Z is ℤ and F(ℝ) ⊂ F(ℤ), but f has no upper bound.
Example 3
Suppose A is regularly closed (i.e. A = $\overline{A°}$ and f is ***lsc*** on X. If f(A°) is upper bounded, then f(b(A)) is upper bounded.

Proof
Let A be regularly closed and f be ***lsc*** on X. Suppose f(b(A)) is has no upper bounded.
Since f (A°) is upper bounded, there exists a real number K such that f (A°) < K.
By assumption, f (b(A)) has an upper bound.
Then, there exists Y ∈b(A) with f(Y) > K.
Let M ∈ℝ such that K < M <f (Y).
Then b(A) is not contained in the closed set $f^{-1}((-\infty, M])$which is a contradiction.
Hence, f (b(A)) is upper bounded.
The following example shows that the condition 'regular closedness' of A, cannot be replaced.
Example 4
Consider A [0,1] ∪ℕ. A is a closed but not a regularly closed subset of ℝ. If f : ℝ⟶ℝ is the identity function, then f f (b(A)) = f (ℕ) has no upper bound. But f (A) is bounded.

Theorem 2
Suppose A is regularly closed and f is ***lsc*** on A. If f has an ***lsc*** extension F to X which is upper bounded on X – A, then f is upper bounded on b(A).

Proof
Let f be the given ***lsc*** on A and A be regularly closed.
Suppose F : X ⟶ ℝ is ***lsc*** which is bounded on X – A.
To show that f is upper bounded on b(A).
Suppose F(X – A) has an upper bound.
Let C $(\overline{X} - \overline{A})$.
Then C is closed such that C° = X – A and so $\overline{C}°= \overline{X} - \overline{A} = C$, which implies that C is regularly closed.
Now b(C) = b $(\overline{X} - \overline{A})$= b (X – A)= b(A).
Hence, b(C) = b(C°) = b(A).
By assumption, F(C°) is upper bounded.
By Lemma, F(b(C)) is upper bounded.
Therefore F(b(A)) is upper bounded.
Hence, f is upper bounded onb(A).

Theorem 3

Let X be a normal space and C a dense subset of the closed set A. Let f be an ***lsc*** function on A. Suppose $^{f}/_{c}$ is continuous and suppose it admits a continuous extension g to A. Then, f has an ***lsc*** extension F to X which is continuous in the points of X – A.

Proof

Let f : A $\rightarrow \mathbb{R}$ be the given ***lsc*** function where A is closed.
Suppose $^{f}/_{c}$ admits a continuous extension g to A.

Then by Tietze's extension theorem, we can consider the continuous extension G of g to X.

Now, define F : X $\rightarrow \mathbb{R}$ by

$$F(X) = \begin{cases} G(X) & if\ X \in X - A \\ f(X) & if\ X \in A \end{cases}$$

To show that F is ***lsc*** on X.

By Theorem 1, g is the lower regulated function of f relative to C and so, for each X in C.
f (X) ≤ g(X)

Let X ∈ A and $\boldsymbol{\varepsilon}$> 0

Since f is ***lsc*** on A, there exists a neighbourhood V of X such that f (z) >f (X) −$\boldsymbol{\varepsilon}$, for each z∈∩A.

Also G is continuous then there exists a neighbourhood T of X with G(Z) > g(X) −$\boldsymbol{\varepsilon}$, for each z∈T.

Let H = T ∩ V.
Then H is a neighbourhood of X.

Case (i)

If z∈ A then Z ∈ V ∩ A.
Therefore by (1), F(z) = f (z) >f (X) −$\boldsymbol{\varepsilon}$.

Case (ii)

If Z ∈ X – A.

Then F(Z) = G(Z) > g(X) −$\boldsymbol{\varepsilon}$
≥f (X) −$\boldsymbol{\varepsilon}$.

Therefore F(z) = f (z) >f (X) −$\boldsymbol{\varepsilon}$, for each z∈ H.
Thus in each case F(z) >f (X)+ $\boldsymbol{\varepsilon}$for each X ∈H.
Hence, F is ***lsc*** on A (a)

Now, Let X ∈ X – A and $\boldsymbol{\varepsilon}$> 0

Then there exists a neighbourhood ∪ of X contained in X – A.
Such that F(Z) ∈ (G(X) −$\boldsymbol{\varepsilon}$, G(X) + $\boldsymbol{\varepsilon}$) for each Z ∈∪ .

Therefore F is continuous at X ∈X – A.
Since X is arbitrary, F is continuous on X – A.

This implies that F is ***lsc*** on X – A (b)

From (a) and (b), F is ***lsc*** on X.

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