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Research Article

# An Interpolation Process on Weighted (0;0,1,2)-Interpolation on Laguerre Polynomial

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**Abstract:** In the present paper ,we have considered the problem in which $\{\xi_i\}_{i=1}^n$ and $\{\xi_i^*\}_{i=1}^n$ the two sets of interscaled nodal points on the interval $[0,\infty)$ . Here we deal with the problem in which function values are prescribed at the zeros of $L_n^k(x)$ and the first derivative values are prescribed on the zeros $L_n^{k-1}(x)$ .We have investigated the existence , uniqueness explicit representation of interpolatory polynomial . Estimation of the fundamental polynomials leading to a convergence theorem have also been obtained.

**Keywords**: Pál - Type interpolation , Laguerre Polynomial, Explicit Representation , Estimation
MSC 2000: 41 A 05 65 D 32.

## Introduction

Pál [10],Mathur P. and Datta S. [8] and many other authors [1][2][6][7] [12] [14] have discussed about various kind of interpolation problems. . In 1975 Pál [10] proved that when the function values are prescribed on one set of n points and derivative values on other set of n-1 points , then there exist no unique polynomial of degree ≤ 2n-2 , but prescribing function value at one more point not belonging to former set of n points there exists a unique polynomial of degree ≤ 2n-1. Lénárd M. [5] investigated the Pál – type interpolation problem on the nodes of Laguerre abscissas.In this paper we study Pál – type interpolational polynomial with $\omega_{n+k}(x) = x^k L_n^{(k)}(x)$.we have examined the existence,
uniqueness ,explicit representation and estimation of fundamental polynomials of such special kind of mixed type of interpolation on interval $[0,\infty)$ . For this we have considered the problem in which $\{\xi_i\}_{i=1}^n$ and $\{\xi_i^*\}_{i=1}^n$ the two sets of interscaled nodal points.

(1.1) $\quad 0 \le \xi_0 < \xi_1^* < \xi_1 < \cdots < \xi_{n-1} < \xi_n^* < \xi_n < \infty$

on the interval $[0,\infty)$ . We seek to determine a polynomial $R_n(x)$ of minimal possible degree 3n+k satisfying the interpolatory conditions :

(1.2) $\quad R_n(\xi_i) = g_i\,,\quad R_n(\xi_i^*) = g_i^*\,,\ R_n'(\xi_i^*) = g_i^{**}\,,\quad R_n''(\xi_i^*) = g_i^{***}\,,\quad for\ i = 1(1)n$

(1.3) $\quad R_n^{(j)}(\xi_0) = g_0^{(j)}\,,\qquad j = 0,1,\ldots,k$

where $g_i$ , $g_i^*$, $g_i^{**}$, $g_i^{***}$ and $g_0^{(j)}$ are arbitrary real numbers. Here Laguerre polynomials $L_n^{(k)}(x)$ and $L_n^{(k-1)}(x)$ have zeroes $\{\xi_i\}_{i=1}^n$ and $\{\xi_i^*\}_{i=1}^n$ respectively and $x_0 = 0$ . We prove existence, uniqueness , explicit representation and estimation of fundamental polynomials.

## Preliminaries

In this section we shall give some well-known results which are as follws :
As we know that the Laguerre polynomial is a constant multiple of a confluent hypergeometric function so the differential equation is given by

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(2.1) $xD^2L_n^k(x)+(1+k-x)DL_n^k(x)+nL_n^k(x)=0$

(2.2) $L_n^{(k-1)'}(x)=-L_{n-1}^{(k)}(x)$

Also using the identities

(2.3) $L_n^{(k)}(x)=L_n^{(k+1)}(x)-L_{n-1}^{(k+1)}(x)$

(2.4) $xL_n^{(k)'}(x)=nL_n^{(k)}(x)-(n+k)L_{n-1}^{(k)}(x)$

We can easily find a relation

(2.5) $\frac{d}{dx}[x^kL_n^k(x)]=(n+k)x^{k-1}L_n^{(k-1)}(x)$

By the following conditions of orthogonality and normalization we define Laguerre polynomial $L_n^{(k)}(x)$,

$for\ k>-1$

(2.6) $\int_0^\infty e^{-x}x^kL_n^{(k)}(x)L_m^{(k)}(x)dx=\Gamma(k+1)\binom{n+k}{n}\delta_{nm}$ $\quad n,m=0,1,2,\dots.$

(2.7) $L_n^{(k)}(x)=\sum_{\mu=0}^{n}\binom{n+k}{n}\frac{(-x)^\mu}{\mu!}$

The fundamental polynomials of Lagrange interpolation are given by

(2.8) $l_j(x)=\frac{L_n^{(k)}(x)}{L_n^{(k)'}(x_j)(x-x_j)}=\delta_{i,j}$

(2.9) $l_j^*(x)=\frac{L_n^{(k-1)}(x)}{L_n^{(k-1)'}(x_j)(x-x_j)}=\delta_{i,j}$

(2.10) $l_j^{*'}(y_j)=\begin{cases}\frac{L_n^{(k-1)'}(y_i)}{L_n^{(k-1)'}(y_j)(y_i-y_j)} & i\neq j\\ -\frac{(k-y_j)}{2y_j} & i=j\end{cases}\quad i,j=1(1)n$

(2.12) $l_j'(y_j)=\frac{1}{(y_j-x_j)}\left[\frac{L_n^{(k)'}(y_j)}{L_n^{(k)'}(x_j)}-\frac{L_n^{(k)}(y_j)}{L_n^{(k)'}(x_j)(y_j-x_j)}\right],\quad j=1(1)n$

For the roots of $L_n^{(k)}(x)$ we have

(2.13) $x_k^2\sim\frac{k^2}{n}$(2.14) $\eta(x)\left|S_n^{(l)}(x)\right|=O(1)$ where $\eta(x)$ is the weight function

(2.15) $\left|L_n^{(k)'}(x_j)\right|\sim j^{-k-\frac{3}{2}}n^{k+1},(0<x_j\le\Omega,n=1,2,3,\dots\dots.)$

(2.16) $\left|L_n^k(x_j)\right|=\begin{cases}x^{-\frac{k}{2}-\frac{1}{4}}O\left(n^{\frac{k}{2}-\frac{1}{4}}\right), & cn^{-1}\le x\le\Omega\\ O(n^k), & 0\le x\le cn^{-1}\end{cases}$

## New Result

**Theorem 1 :** For n and k fixed positive integer let $\{g_i\}_{i=1}^n$ , $\{g_i^*\}_{i=1}^n$ , $\{g_i^{**}\}_{i=1}^n$, $\{g_i^{***}\}_{i=1}^n$ and , $\{g_0^{(j)}\}_{j=0}^k$ are arbitrary real numbers then there exists a unique polynomial $R_n(x)$ of minimal possible degree ≤ 4n+k on the nodal points (1.1) satisfying the condition (1.2) and (1.3). The polynomial $R_n(x)$ can be written in the form

(3.1) $$\boldsymbol{R_n(x) = \sum_{j=1}^n U_j(x)g_j + \sum_{j=1}^n V_j(x)g_j^* + \sum_{j=1}^n W_j(x)g_j^{**} + \sum_{j=1}^n X_j(x)g_j^{***} + \sum_{j=0}^k C_j(x)g_0^{(j)}}$$

where $U_j(x)$ , $V_j(x)$ , $W_j(x)$ , $X_j(x)$ and $C_j(x)$ are fundamental polynomials of degree ≤ 4n+k given by

(3.2) $$U_j(x) = \frac{x^{(k+1)}\ l_j(x)\ [L_n^{(k-1)}(x)]^3}{x_j^{(k+1)}[L_n^{(k-1)}(x_j)]^3}$$

(3.3) $$V_j(x) = \frac{x^{k+1}\{l_j^*(x)\}^3 L_n^k(x)}{y_j^{(k+1)} L_n^{(k)}(y_j)}\left[1-(x-y_j)\left\{\frac{(3y_j-3k+2)}{2y_j}+(\sigma_1+\sigma_2)(x-y_j)\right\}\right]$$

(3.4) $$W_j(x) = \frac{x^{k+1}\ [l_j^*(x)]^2\ L_n^{(k)}(x)\ \ L_n^{(k-1)}(x)\ [y_j+(k-y_j)(x-y_j)]}{y_j^{k+2}\ L_n^k(y_j)\ \ L_n^{(k-1)'}(y_j)}$$

(3.5) $$X_j(x) = \frac{e^{y_j/2}x^{k+1}\ l_j^*(x)\ L_n^{(k)}(x)\ \ [L_n^{(k-1)}(x)]^2}{y_j^{\frac{3k}{2}+1}\ L_n^{(k)}(y_j)\ [L_n^{(k-1)'}(y_j)]^2}$$

(3.6) $$C_j(x) = p_j(x)x^j[L_n^k(x)]^2[L_n^{(k-1)}(x)]^2 + x^k L_n^{(k)}(x)\ [L_n^{(k-1)}(x)]^2\left[c_j^* - \frac{L_n^k(x)p_j(x)+q_j(x)L_n^{(k-1)}(x)}{x^{k-j}}\right],$$

$,j = 0,1,\dots,k-1$

(3.7) $$C_k(x) = \frac{1}{k!L_n^k(0)[L_n^{(k-1)}(0)]^3}x^k L_n^{(k)}(x)[L_n^{(k-1)}(x)]^{\,3}$$

where $p_j(x)$ and $q_j(x)$ are polynomials of degree at most k-j-1 . $c_j$ is defined in (4.14)

**Theorem 2** Let the interpolatory function $f: R \to R$ be continuously differentiable such that,
$C(m) = \{f(x): f\ is\ continuous\ in[0,\infty), f(x) = 0(x^m) as\ x \to \infty; m \geq 0\ is\ an\ integer\}$
For every $f \in C(m)$ and $\alpha \geq 0$ , Then

(3.8) $$R_n(x) = \sum_{j=1}^n U_j(x)g_j + \sum_{j=1}^n V_j(x)g_j^* + \sum_{j=1}^n W_j(x)g_j^{**} + \sum_{j=1}^n X_j(x)g_j^{***} + \sum_{j=0}^k C_j(x)g_0^{(j)}$$

satisfies the relation

(3.9) $$|R_n(x) - f(x)| = 0\left(n^{\frac{k}{2}-2}\right)\omega\left(f, \frac{\log n}{\sqrt{n}}\right) , \qquad \text{for } 0 \leq x \leq cn^{-1}$$

(3.10) $$|R_n(x) - f(x)| = 0\left(n^{\frac{k}{2}}\right)\omega\left(f, \frac{\log n}{\sqrt{n}}\right), \qquad \text{for } cn^{-1} \leq x \leq \Omega$$

where ω is the modulus of continuity.

**Proof of Theorem 1**

Let $U_j(x)$ , $V_j(x)$ , $W_j(x)$, $X_j(x)$ and $C_j(x)$ are polynomials of degree ≤ 4n+k satisfying conditions (4.1) , (4.2) , (4.3), (4.4) and (4.5) respectively.

$$(4.1)\quad \begin{cases} U_j(x_i) = \begin{cases} 0 & for\ i\neq j \\ 1 & i=j \end{cases}, \quad U_j(y_i) = 0, \quad U_j'(y_i) = 0 \\ and \\ [\rho(x)U_j(x)]''_{x=y_j} = 0, \quad U_j^{(l)}(0) = 0, \end{cases}$$

$$i = 1(1)n \ \ and \quad l = 0,1,\dots,k$$

$$(4.2)\quad \begin{cases} V_j(x_i) = 0, \quad V_j(y_i) = \begin{cases} 0 & for\ i\neq j \\ 1 & i=j \end{cases}, \quad V_j'(y_i) = 0 \\ and \\ [\rho(x)V_j(x)]''_{x=y_j} = 0, \quad V_j^{(l)}(0) = 0, \end{cases}$$

$$i = 1(1)n \ \ and \quad l = 0,1,\dots,k$$

$$(4.3)\quad \begin{cases} W_j(x_i) = 0, \quad W_j(y_i) = 0, \quad W_j'(y_i) = \begin{cases} 0 & for\ i\neq j \\ 1 & i=j \end{cases} \\ and \\ \left[\rho(x)W_j(x)\right]''_{x=y_j} = 0, \quad W_j^{(l)}(0) = 0, \end{cases}$$

$$i = 1(1)n \ \ and \quad l = 0,1,\dots,k$$

$$(4.4)\quad \begin{cases} X_j(x_i) = 0, \quad X_j(y_i) = 0, \quad X_j'(y_i) = 0, \\ and \\ [\rho(x)X_j(x)]''_{x=y_j} = \begin{cases} 0 & for\ i\neq j \\ 1 & i=j \end{cases}, \quad X_j^{(l)}(0) = 0, \end{cases}$$

$$i = 1(1)n \ \ and \quad l = 0,1,\dots,k$$

$$(4.5)\quad \begin{cases} C_k(x_i) = 0, \quad C_k(y_i) = 0, \quad C_k'(y_i) = 0, \\ and \\ [\rho(x)C_k(x)]''_{x=y_j} = 0 \quad C_k^{(l)}(0) = \begin{cases} 0 & for\ i\neq j \\ 1 & i=j \end{cases}, \quad i = 1(1)n \end{cases}$$

$$i = 1(1)n \ \ and \quad l = 0,1,\dots,k$$

To determine $U_j(x)$ let

(4.6) $\quad U_j(x) = C_1 x^{k+1} l_j(x) [L_n^{(k-1)}(x)]^3$

where $C_1$is a constant . $l_j(x)$ is defined in (2.8) . $U_j(x)$ is a polynomial of degree ≤ 4n+k By using (2.8) we determine

(4.7) $$C_1 == \frac{1}{x_j^{(k+1)}[L_n^{(k-1)}(x_j)]^3}$$

Hence we find the first fundamental polynomial $U_j(x)$ of degree ≤ 4n+k

To find second fundamental polynomial let

( 4.8) $$V_j(x) = x^{k+1}[l_j^*(x)]^3 L_n^{(k)}(x)[C_2 + C_3(x - y_j)]$$

$$+C_4 x^{k+1} l_j^*(x) L_n^{(k)}(x)[L_n^{(k-1)}(x)]^2$$

where $C_2$ , $C_3$ and $C_4$ are arbitrary constants. By using (2.9) and (4.2) we determine

(4.9) $$C_2 = \frac{1}{y_j^{(k+1)} L_n^k(y_j)} ,$$

(4.10) $$C_3 == -\frac{3y_j - 3k + 2}{2y_j} C_2$$ and

(4.11) $$C_4 = -\frac{\{\sigma_1 + \sigma_2\}}{[L_n^{(k-1)'}(y_j)]^2} C_2$$

where $$\sigma_1 = \frac{(3y_j - 3k + 2)}{2{y_j}^2}[\tfrac{1}{2}(y_j + k) + \tfrac{y_j}{2}(3y_j - 5k) + 2] ,$$

and $$\sigma_2 = \frac{1}{{y_j}^2}[\tfrac{3}{4}(y_j + k)^2 - \tfrac{y_j}{2}(4n - 5) - 2k] ,$$

Hence we find the first fundamental polynomial $V_j(x)$ of degree ≤ 4n+k Again let

(4.12) $$W_j(x) = x^{k+1} L_n^{(k)}(x) L_n^{(k-1)}(x) l_j^*(x)[C_5 l_j^*(x) + C_3 L_n^{(k-1)}(x)]$$

Where $C_5$ and $C_6$ are arbitrary constants , $l_j^*(x)$ is defined in (2.9) . $W_j(x)$ is polynomial of degree ≤ 4n+k satisfying the conditions (4.3) by which we obtain

(4.13) $$C_5 = \frac{1}{y_j^{k+1} L_n^k(y_j) L_n^{(k-1)'}(y_j)}$$ and

(4.14) $$C_6 = \frac{(k - y_j)}{y_i^{k+2}\, L_n^{(k)}(y_j)\, [L_n^{(k-1)'}(y_j)]^2}$$

Hence we find the third fundamental polynomial $W_j(x)$ of degree ≤ 4n+k Again let

(4.15) $$X_j(x) = C_7 x^{k+1} l_j^*(x) L_n^{(k)}(x)[L_n^{(k-1)}(x)]^2$$

where $C_7$ is a constant , $l_j^*(x)$ is defined in (2.9) . $X_j(x)$ is polynomial of degree ≤ 4n+k satisfying the conditions (4.4) by which we obtain

(4.16) $$C_7 = \frac{1}{y_j^{\frac{3k}{2}+1}\, L_n^{(k)}(y_j)\, [L_n^{(k-1)'}(y_j)]^2}$$

Hence we find the third fundamental polynomial $X_j(x)$ of degree ≤ 4n+k

To find $C_j(x)$ , we assume $C_j(x)$ for fixed $j \epsilon \{0,1, \dots \dots \dots, k-1\}$

(4.17) $$C_j(x) = p_j(x) x^j [L_n^k(x)]^2 [L_n^{(k-1)}(x)]^2 + x^k L_n^{(k)}(x)[L_n^{(k-1)}(x)]^2 g_n(x)$$

where $p_j(x)$ and $g_n(x)$ are polynomials of degree k-j-1 and n respectively . Now it is clear that

$C_j^{(l)}(0) = 0 \quad for \quad (l = 0, \dots\dots, j-1)$ and since $L_n^{(k)}(x_i) = 0$ and $L_n^{(k-1)}(y_i) = 0$ we get $C_j(x_i) = 0$ *and* $C_j(y_i) = 0$ *for* $i = 1(1)n$ . The coefficient of the polynomial $p_j(x)$ are calculated by the system

(4.18) $$C_j^{(l)}(0) = \frac{d^l}{dx^l}\left[p_j(x)x^j[L_n^k(x)]^2[L_n^{(k-1)}(x)]^2\right]_{x=0} = \delta_{i,j} \qquad (l = j, \dots\dots, k-1)$$

Now using the condition $[\rho(x)C_k^*(x)]''_{x=y_j} = 0$

we get (4.19) $g_n(y_i) = -(y_i)^{j-k}L_n^k(y_i)p_j(y_i)$ , which implies $g_n(x)$ as follows

(4.20) $$g_n(x) = -\frac{L_n^k(x)p_j^*(x)+q_j^*(x)L_n^{(k-1)}(x)}{x^{k-j}}$$

where $q_j(x)$ is a polynomial of degree k-j .

Using (4.17) and (4.20) we obtain $C_j(x)$ of degree ≤ 4n+k satisfying the conditions (4.6)

**Estimation Of The Fundamental Polynomials**

**Lemma 5.1.** Let *the fundamental polynomial* $U_j(x)$ , *for* $j = 1,2, \dots, n$ *be given by* (3.2), *then we have*

(5.1) $$\sum_{j=1}^{n} e^{x_j/2}x_j^{-k/2}\left|U_j(x)\right| = 0\left(n^{-\frac{k}{2}}\right), \qquad \text{for } 0 \le x \le cn^{-1}$$

(5.2) $$\sum_{j=1}^{n} e^{x_j/2}x_j^{-k/2}\left|U_j(x)\right| = 0\left(n^{-\frac{k}{2}}\right), \text{for } cn^{-1} \le x \le \Omega$$

where $U_j(x)$ is given in equqtion (3.2) Proof : From (3.2) we have

(5.3) $$\sum_{j=1}^{n} e^{x_j/2}x_j^{-k/2}\left|U_j(x)\right|$$
$$\le \sum_{j=1}^{n}\frac{e^{x_j/2}x_j^{-k/2}|x^{(k+1)}|\ |l_j(x)|\ [L_n^{(k-1)}(x)]^3}{\left|x_j^{(k+1)}\right|\ [L_n^{(k-1)}(x_j)]^3},$$

where $U_j(x)$ is given in equqtion (3.2)

Thus by (5.3) and (2.16) , we get the result.

**Lemma 5.2** Let *the fundamental polynomial* $V_j(x)$ , *for* $j = 1,2, \dots, n$ *be given by* (3.3) , *then we have*

(5.4) $$\sum_{j=1}^{n} e^{y_j/2}y_j^{-k/2}\left|V_j(x)\right| = 0\left(n^{-\frac{k}{2}}\right), \qquad \text{for } 0 \le x \le cn^{-1}$$

(5.5) $$\sum_{j=1}^{n} e^{y_j/2}y_j^{-k/2}\left|V_j(x)\right| = 0\left(n^{-\frac{k}{2}}\right), \text{for } cn^{-1} \le x \le \Omega$$

where $V_j(x)$ is given in equation (5.3).

Proof : From (3.3) we have

$$\left|V_j(x)\right| \le \frac{|x^{k+1}|\{l_j^*(x)\}^3|L_n^k(x)|}{y_j^{(k+1)}\left|L_n^{(k)}(y_j)\right|}$$
$$+\frac{|x^{k+1}|\{l_j^*(x)\}^3(x-y_j)L_n^k(x)(3y_j-3k+2)}{2y_jy_j^{(k+1)}L_n^{(k)}(y_j)}$$
$$+\frac{|x^{k+1}|\{l_j^*(x)\}^3|L_n^k(x)|\ |\sigma_1+\sigma_2)|\ (x-y_j)^2}{2\ \left|y_j^{(k+1)}\right|\left|L_n^{(k)}(y_j)\right|}$$

(5.6) $$\sum_{j=1}^{n} e^{y_j/2}y_j^{-k/2}\left|V_j(x)\right|$$
$$\le \sum_{j=1}^{n}\frac{e^{y_j/2}y_j^{-k/2}|x^{k+1}|\{l_j^*(x)\}^3|L_n^k(x)|}{\left|y_j^{(k+1)}\right|\left|L_n^{(k)}(y_j)\right|}$$
$$+\sum_{j=1}^{n}\frac{e^{y_j/2}y_j^{-k/2}\ |x^{k+1}|\{l_j^*(x)\}^3(x-y_j)|L_n^k(x)||3y_j-3k+2|}{2|y_j|\left|y_j^{(k+1)}\right|L_n^{(k)}(y_j)}$$
$$+\sum_{j=1}^{n}\frac{e^{y_j/2}y_j^{-k/2}|x^{k+1}|\{l_j^*(x)\}^3|L_n^k(x)|\ |\sigma_1+\sigma_2)|\ (x-y_j)^2}{2\ \left|y_j^{(k+1)}\right|\left|L_n^{(k)}(y_j)\right|}$$

$= \zeta_1 + \zeta_2 + \zeta_3$

where

$$\zeta_1 = \sum_{j=1}^{n} \frac{e^{y_j/2} y_j^{-k/2} |x^{k+1}| \{l_j^*(x)\}^3 |L_n^k(x)|}{|y_j^{(k+1)}| |L_n^{(k)}(y_j)|}$$

$$\zeta_2 = \sum_{j=1}^{n} \frac{e^{y_j/2} y_j^{-k/2} \ |x^{k+1}| \{l_j^*(x)\}^3 (x-y_j) |L_n^k(x)| |3y_j-3k+2|}{2|y_j| |y_j^{(k+1)}| L_n^{(k)}(y_j)}$$

$$\zeta_3 = \sum_{j=1}^{n} \frac{e^{y_j/2} y_j^{-k/2} |x^{k+1}| \{l_j^*(x)\}^3 |L_n^k(x)| \ |\sigma_1+\sigma_2| \ (x-y_j)^2}{2 \ |y_j^{(k+1)}| |L_n^{(k)}(y_j)|}$$

Thus by using (5.6) and (2.16) , we yield the result.

**Lemma 5.3** Let *the fundamental polynomial* $W_j(x)$ , *for* $j = 1,2, \ldots, n$ *given by* (3.4) , *then we have*

(5.7) $$\sum_{j=1}^{n} e^{y_j/2} y_j^{-k/2} |W_j(x)| = O\left(n^{-\frac{k}{2}}\right), \qquad \text{for } 0 \le x \le cn^{-1}$$

(5.8) $$\sum_{j=1}^{n} e^{y_j/2} y_j^{-k/2} |W_j(x)| = O\left(n^{-\frac{k}{2}}\right), \text{for } cn^{-1} \le x \le \Omega$$

where $W_j(x)$ is given in equation (3.4)

Proof : From (3.4) we have

$$|W_j(x)| \le \frac{|x^{k+1}| \ [l_j^*(x)]^2 \ |L_n^{(k)}(x)| \ | \ L_n^{(k-1)}(x)| \ |y_j|}{|y_j^{k+2}| \ [L_n^k(y_j)]^2} + $$
$$+ \ \frac{|x^{k+1}| \ l_j^*(x) \ |L_n^k(x)| \ |k-y_j| \ |L_n^{k-1}(x)|}{|y_j^{k+2}| \ [L_n^{(k)}(y_j)]^3}$$

(5.9) $$\sum_{j=1}^{n} e^{y_j/2} y_j^{-k/2} |W_j(x)|$$

$$\le \sum_{j=1}^{n} \frac{e^{y_j/2} y_j^{-k/2} |x^{k+1}| \ [l_j^*(x)]^2 \ |L_n^{(k)}(x)| \ | \ L_n^{(k-1)}(x)| \ |y_j|}{|y_j^{k+2}| \ [L_n^k(y_j)^2]}$$

$$+ \ \sum_{j=1}^{n} \frac{e^{y_j/2} y_j^{-k/2} |x^{k+1}| \ l_j^*(x) \ |L_n^k(x)| \ |k-y_j| \ |L_n^{k-1}(x)|}{|y_j^{k+2}| \ [L_n^{(k)}(y_j)]^3}$$

$$= \zeta_4 + \zeta_5$$

where

$$\zeta_4 = \sum_{j=1}^{n} \frac{e^{y_j/2} y_j^{-k/2} |x^{k+1}| \ [l_j^*(x)]^2 \ |L_n^{(k)}(x)| \ | \ L_n^{(k-1)}(x)| \ |y_j|}{|y_j^{k+2}| \ [L_n^k(y_j)^2]}$$

$$\zeta_5 = \sum_{j=1}^{n} \frac{e^{y_j/2} y_j^{-k/2} |x^{k+1}| \ l_j^*(x) \ |L_n^k(x)| \ |k-y_j| \ |L_n^{k-1}(x)|}{|y_j^{k+2}| \ [L_n^{(k)}(y_j)]^3}$$

Thus by using (5.9) and (2.16) , we get the result.

**Lemma 5.3.4** Let *the fundamental polynomial* $X_j(x)$ , *for* $j = 1,2, \ldots, n$ *be given by* (3.5) , *then we have*

(5.10) $$\sum_{j=1}^{n} |X_j(x)| = O\left(n^{-\frac{k}{2}-2}\right), \qquad \text{for } 0 \le x \le cn^{-1}$$

(5.11) $$\sum_{j=1}^{n} |X_j(x)| = O\left(n^{-\frac{k}{2}}\right), \qquad \text{for } cn^{-1} \le x \le \Omega$$

where $X_j(x)$ is given in equqtion (3.5) .

Proof : From (3.5) we have

$$(5.12) \qquad \sum_{j=1}^{n}\left|X_j(x)\right| \le \sum_{i=1}^{n} \frac{e^{y_j/2}\left|\, x^{k+1}\right|\, l_j^*(x)\, \left|L_n^{(k)}(x)\right|\, [L_n^{(k-1)}(x)]^2}{\left|y_j^{\frac{3k}{2}+1}\right| \left|L_n^{(k)}(y_j)\right| [L_n^{(k-1)'}(y_j)]^2}$$

By equations (5.12) and (2.16) we yield the result.
Now we state our main theorem in § 6.

**Proof Of Theorem 2**

We prove theorem 2 with the help of certain theorem mentioned as below :

**Theorem :** Let $C(m) = \{f(x)\text{: } f \text{ is continuous in}[0,\infty),\ f(x) = O(x^m) \text{as } x \to \infty;\ m \ge 0 \text{ is an integer}\}$ Then by Szego[12] is

$$\lim_{n\to\infty}\left\|f(x) - H_n^{(\alpha)}(f,x)\right\|_I = 0$$

For every $f \in C(s)$ and $I \subset (0,\infty)$ $for\ \alpha \ge 0$ , or $I \subset (0,\infty)$ $for -1 < \alpha < 0$ . furthermore there exists a function in C(m) such that $\{H_n^{(\alpha)}(f,x)\}$ diverges for $\alpha \ge 0$ at x=0 .As for thr rate of convergence the following result is due to Vertesi [15]

$$(6.1) \qquad \left\|f(x) - H_n^{(\alpha)}(f,x)\right\|_I = \begin{cases} O(\omega(f, n^{-1-\alpha})); & -1 < \alpha < \\ O\left(\omega\left(f, \frac{\log n}{\sqrt{n}}\right)\right); & \alpha \ge -\frac{1}{2}0 \end{cases}$$

**Proof of main theorem 2:**

Since $R_n(x)$ given by equation (5.2.1) is exact for all polynomial $Q_n(x)$ of degree ≤ 4n+k , we have

$$(6.1) \quad Q_n(x) = \sum_{j=1}^{n} Q_n(x_j)U_j(x) + \sum_{j=1}^{n} Q_n(y_j)V_j(x) + \sum_{j=1}^{n} Q_n{}'(y_j)W_j(x) + \sum_{j=1}^{n}[\rho(x)Q_n(x)]''_{x=y_j} X_j(x) + \sum_{j=0}^{k} Q_n(x_0)C_j(x)$$

From equation (5.2.1) and (5.5.1) we get

$$(6.2) \quad \rho(x)|f(x) - R_n(x)| \le \rho(x)|f(x) - Q_n(x)| + \rho(x)|Q_n(x) - R_n(x)|$$
$$\le \rho(x)|f(x) - Q_n(x)| + \sum_{j=1}^{n} \rho(x)\left|f(x_j) - Q_n(x_j)\right| U_j(x)$$
$$+ \sum_{j=1}^{n} \rho(x)\left|f(y_j) - Q_n(y_j)\right| |V_j(x)|$$
$$+ \sum_{j=1}^{n} \rho(x)\left|f'(y_j) - Q_n{}'(y_j)\right| |W_j(x)|$$
$$+ \sum_{j=1}^{n}[\rho(x)Q_n(x)]''_{x=y_j} |X_j(x)|$$
$$+ \sum_{j=0}^{k} \rho(x)\left|f^{l}(x_0) - Q_n{}^{l}(x_0)\right| |C_j(x)|$$

Thus (6.2) and Lemmas 5.3.1 , 5.3.2, 5.3.3 and 5.3.4 completes the proof of the theorem.

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