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# Extension of the $\delta$ - Function of $\mathbb{R}^n$

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#### Abstract

The delta function plays a vital role in many areas of mathematics. Our objective in this paper is to extend it to higher dimensional spaces and to study some of its fundamental properties.

#### 1. The $\delta$ - function

# 1.1 Definition:

Let  $\mathbb{R}$  be the set of real numbers and  $\mathbb{C}$  be the set of complex numbers.

The  $\boldsymbol{\delta}$  - function on a subset E of  $\mathbb R$  or  $\mathbb C$  is the function

 $\delta : E \rightarrow \{0, 1\}$  defined by

$$\delta(x) = \begin{cases} 0 \ if \ x = 0\\ 1 \ if \ x \neq 0 \end{cases}$$
(1)

The first thing we observe is that  $\delta$  is a mininorm on  $X = \mathbb{R}$  or  $\mathbb{C}$ . Before writing a proof for this simple observation, let us define a mininorm.

### 1.2 Definition

Let X be a vector space over  $K = \mathbb{R}$  or  $\mathbb{C}$ . A mininorm on X is a function  $w = X \to \mathbb{R}$  which satisfy the following conditions:

(a) $w(x) \ge 0$ for all $x \in X$	(2)
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(b) $w(\alpha x) = w(x)$ for all $x \in X$ and $\alpha \in K$ , $0 \neq \alpha \in K$	(4)
(c) $w(x + y) \le w(x) + w(y)$ for all $x, y \in X$	(5)

A vector space X with a mininorm w defined on it is called a mininormed space and is, in general, denoted by (X, w).

**Note:** Every mininorm w induces a metric  $d_w$  defined by

(6)

 $d_w(x, y) = w(x - y)$  for all  $x, y \in X$ 

## **1.3 Proposition**

Let  $X = \mathbb{R}$  or  $\mathbb{C}$ . Then  $\boldsymbol{\delta}$  is a mininorm on X.

#### **Proof** :

Condition (a) for a mininorm is obvious from the definition of  $\delta$ .

To verify (b) take  $x \in X$  and  $\alpha \in K$  with  $\alpha \neq 0$ .

If x = 0, then  $\alpha x = 0$  so that  $\delta(x) = \delta(\alpha x) = 0$ .

If  $x \neq 0$ , then  $\alpha x \neq 0$  so that  $\delta(x) = \delta(\alpha x) = 1$ .

Now, we prove (c).

Suppose x + y = 0. Then (c) is obvious.

Now suppose  $x + y \neq 0$ . Then, at least one of x and y is non zero. Without loss of generality, we may assume that  $x \neq 0$ . Then,  $\delta(x) = 1$ 

So,  $\delta(x) + \delta(y) \ge 1$ . But  $\delta(x + y) = 1$ .

Hence  $\delta(x + y) \leq \delta(x) + \delta(y)$ .

#### 1.4 Remark:

It can be easily checked that whenever w is a mininorm on  $\mathbb{R}$  or  $\mathbb{C}$ , rw is also a mininorm on  $\mathbb{R}$  or  $\mathbb{C}$  where r is any real number  $r \neq 0$ . Hence, for any real number  $r \neq 0$ ,  $r\delta$  is also a mininorm on  $X \in \mathbb{R}$  or  $\mathbb{C} \cdot r\delta$  is the function given by

$$r\delta(x) = \begin{cases} 0 & \text{if } x = 0\\ 1 & \text{if } x \neq 0 \end{cases}$$

$$\tag{7}$$

we may denote  $r\delta$  by  $\delta_r$ .

### **2.** Extension of the $\delta$ - Function of $\mathbb{R}^n$

#### 2.1 Definition:

Let  $X = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ 

Define  $\delta(x)$  by

$$\delta(x) = (\delta(x_1), \delta(x_2), \dots, \delta(x_n)) \qquad (8)$$

Let us now define an order relation on  $\mathbb{R}^n$ .

### 2.2 Definition:

Let  $X = (x_1, x_2, ..., x_n)$  and  $Y = (y_1, y_2, ..., y_n) \in \mathbb{R}^n$  We say that  $x \le y$  if and only if  $x_i \le y_i$  for all i = 1, 2, 3, ..., n. It is clear that  $\le$  is a partial order relation on  $\mathbb{R}^n$ .

As an illustration,  $(2, 0, -3) \leq (5, 1, 0)$  in  $\mathbb{R}^3$ .

But the vectors (2, 0, -3) and (5, 1, 0) are not comparable with respect to this order. Thus, the law of trichotomy does not hold in  $\mathbb{R}^n$  with respect to this order for n > 1.

It is now interesting to note that most of the properties of  $\delta$  on  $\mathbb{R}$  hold for the extended  $\delta$  on  $\mathbb{R}^n$ .

**2.3 Proposition:**  $\delta$  in  $\mathbb{R}^n$  satisfies the following:

(a)  $\delta(x) \ge 0$  for all  $x \in \mathbb{R}^n$  and  $\delta(x) = 0$  if and only if x = 0.

(b)  $\delta(rx) = \delta(x)$  for all  $x \in \mathbb{R}^n$  and for all  $0 \neq r \in \mathbb{R}$ .

(c)  $\delta(x + y) \leq \delta(x) + \delta(y)$  for all  $x, y \in \mathbb{R}^n$ .

# **Proof:**

Let  $X = (x_1, x_2, ..., x_n)$  and  $Y = (y_1, y_2, ..., y_n) \in \mathbb{R}^n$ 

Then, clearly  $\delta(x) = (\delta(x_1), \delta(x_2), \dots, \delta(x_n)) \ge 0$ .

since  $\delta(x_i) \ge 0$  for all *i*.

And  $\delta(x) = 0$  if and only if  $\delta(x_i) = 0$  for all *i*.

if and only if  $x_i = 0$  for all *i* if and only if x = 0.

For  $r \neq 0$ , consider

$$\delta (rx) = (\delta rx_1, rx_2, \dots, rx_n)$$
  
$$\delta (x) = (\delta (rx_1), \delta (rx_2), \dots, \delta (rx_n))$$
  
$$\delta (x) = (\delta (x_1), \delta (x_2), \dots, \delta (x_n)) = \delta (x)$$

Further,

$$\delta (x + y) = \delta (x_1 + y_1, x_2 + y_2, ..., x_n + y_n)$$
  
=  $(\delta (x_1 + y_1), \delta (x_2 + y_2), ..., \delta (x_n + y_n))$  (9)

Now,  $\delta(x_i + y_i) \leq \delta(x_i) + \delta(y_i)$  for all *i*.

So, (9) gives

$$\delta(x + y) = (\delta(x_1) + \delta(y_1)), (\delta(x_2) + \delta(y_2), ..., (\delta(x_n) + \delta(n))),$$
  
=  $(\delta(x_1), \delta(x_2), ..., \delta(x_n)) + (\delta(y), \delta(y_2), ..., \delta(y_n))$ 

For an  $X = (x_1, x_2, ..., x_n)$  in  $\mathbb{R}^n$ , its norm ||x|| is defined by

$$\|x\| = (x_1^2 + x_2^2 + \dots + x_n^2)^{1/2}.$$
 (10) For

$$x = (x_1, x_2, \dots, x_n), \text{ put } |x| = (|x_1|, |x_2|, \dots, |x_n|).$$
(11)

Also, for  $r \in \mathbb{R}^n$ , put  $\bar{r} = (r, r, ..., r) \in \mathbb{R}^n$ . (12)

For example,  $\overline{1} = (1, 1, \dots, 1)$ .

Now we have the result:

# 2.4 Proposition

Let  $x \in \mathbb{R}^n$ .

(a) If  $x \ge 1$ , then  $||x|| \ge ||\delta(x)||$ 

(b) If  $|x| \le 1$ , then  $||x|| \le ||\delta(x)||$ .

#### **Proof:**

Suppose  $x \ge 1$ . That is,  $x_i \ge 1$  for all *i*.

So,  $x_i^2 \ge 1^2 = \delta (x_i)^2$  for all *i*.

Hence, 
$$x_1^2 + x_2^2 + \dots + x_n^2 \ge \delta(x_1)^2 + \delta(x_2)^2 + \dots + \delta(x_n)^2$$

which implies

 $\|x\| \ge \|\delta(x)\|$ 

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Now, if |x| \le 1, then |x_i| \le 1, for all i.
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so that  $x_i^2 \le 1^2 = \delta (x_i)^2$ 

Hence we get,  $||x|| \le ||\delta(x)||$ 

#### Note:

It is not true that  $x \le 1$  implies  $||x|| \le ||\delta(x)||$ 

For example, let X = (-2, 0, 0) in  $\mathbb{R}^3$ 

Then  $x \leq 1$ .

 $\delta(x) = (1, 0, 0) \text{ and } \|\delta(x)\| = 1.$ 

But  $||x|| = 2 \ge ||\delta(x)||$ .

# 2.5 Definition:

For i = 1, 2, ..., n,  $e_i$  is the vector defined by

 $e_i = (0,0, ..., 1, 0, ..., 0)$ , with 1 occurs in the  $i^{th}$  place and all other co ordinators are 0.

# **Remark:**

 $\delta(e_i) = e_i$ 

More generally, for  $r \neq 0$ ,

$$\begin{split} \delta(re_i) &= \ \delta \ (0,0,\ldots,r,0,0,\ldots,0) \\ &= \ (0,0,\ldots,\delta(r),0,0,\ldots,0) \\ &= \ (0,0,\ldots,1,0,0,\ldots,0) = e_i \end{split}$$

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