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# Extension of the $\delta$ - Function of $\mathbb{R}^n$

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**Abstract**

The delta function plays a vital role in many areas of mathematics. Our objective in this paper is to extend it to higher dimensional spaces and to study some of its fundamental properties.

## 1. The $\delta$ - function

### 1.1 Definition:

Let $\mathbb{R}$ be the set of real numbers and $\mathbb{C}$ be the set of complex numbers.

The $\boldsymbol{\delta}$ - function on a subset E of $\mathbb{R}$ or $\mathbb{C}$ is the function

$\boldsymbol{\delta} : \boldsymbol{E} \rightarrow \{\mathbf{0}, \mathbf{1}\}$ defined by

$$\delta(x) = \begin{Bmatrix} 0 \ if \ x = 0 \\ 1 \ if \ x \neq 0 \end{Bmatrix} \quad (1)$$

The first thing we observe is that $\boldsymbol{\delta}$ is a mininorm on $X = \mathbb{R} \ or \ \mathbb{C}$ .Before writing a proof for this simple observation, let us define a mininorm.

### 1.2 Definition

Let $X$ be a vector space over $K = \mathbb{R} \ or \ \mathbb{C}$ . A mininorm on $X$ is a function $w = X \rightarrow \mathbb{R}$ which satisfy the following conditions:

(a) $w(x) \geq 0$ for all $x \in X$ (2)

and $w(x) = 0$ if and only if $x = 0$ (3)

(b) $w(\alpha x) = w(x)$ for all $x \in X$ and $\alpha \in K$ , $0 \neq \alpha \in K$ (4)

(c) $w(x + y) \leq w(x) + w(y)$ for all $x, y \in X$ (5)

A vector space $X$ with a mininorm $w$ defined on it is called a mininormed space and is, in general, denoted by $(X, w)$.

**Note:** Every mininorm $w$ induces a metric $d_w$ defined by

$d_w(x, y) = w\,(x - y)$ for all $x, y \in X$ (6)

**1.3 Proposition**

Let $X = \mathbb{R}\ or\ \mathbb{C}$ . Then $\boldsymbol{\delta}$ is a mininorm on $X$.

**Proof :**

Condition (a) for a mininorm is obvious from the definition of $\boldsymbol{\delta}$.

To verify (b) take $x \in X$ and $\alpha \in K$ with $\alpha \neq 0$.

If $x = 0$ , then $\alpha x = 0$ so that $\delta\,(x) = \delta\,(\alpha x) = 0$.

If $x \neq 0$, then $\alpha x \neq 0$ so that $\delta\,(x) = \delta\,(\alpha x) = 1$.

Now, we prove (c).

Suppose $x + y = 0$. Then (c) is obvious.

Now suppose $x + y \neq 0$. Then, atleast one of $x$ and $y$ is non zero. Without loss of generality, we may assume that $x \neq 0$ . *Then,* $\delta\,(x) = 1$

*So,* $\delta\,(x) + \delta\,(y) \geq 1$ . But $\delta\,(x + y) = 1$ .

Hence $\delta\,(x + y) \leq \delta\,(x) + \delta\,(y)$.

**1.4 Remark:**

It can be easily checked that whenever $w$ is a mininorm on $\mathbb{R}\ or\ \mathbb{C}$ , $rw$ is also a mininorm on $\mathbb{R}\ or\ \mathbb{C}$ where $r$ is any real number $r \neq 0$. Hence, for any real number $r \neq 0$, $r\delta$ is also a mininorm on $X \in \mathbb{R}\ or\ \mathbb{C}$ . $r\delta$ is the function given by

$$r\delta(x) = \begin{cases} 0\ if\ x = 0 \\ 1\ if\ x \neq 0 \end{cases} \tag{7}$$

we may denote $r\delta$ by $\delta_r$.

**2. Extension of the $\boldsymbol{\delta}$ - Function of $\mathbb{R}^n$**

**2.1 Definition:**

Let $X = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$

Define $\delta\,(x)$ by

$$\delta\,(x) = (\delta\,(x_1), \delta\,(x_2), \ldots, \delta\,(x_n)) \quad . \tag{8}$$

Let us now define an order relation on $\mathbb{R}^n$.

**2.2 Definition:**

Let $X = (x_1, x_2, \dots, x_n)$ *and* $Y = (y_1, y_2, \dots, y_n) \epsilon \mathbb{R}^n$ We say that $x \le y$ if and only if $x_i \le y_i$ for all $i = 1, 2, 3, \dots, n$. It is clear that $\le$ is a partial order relation on$\mathbb{R}^n$.

As an illustration, $(2, 0, -3) \le (5, 1, 0) in\ \mathbb{R}^3$ .

But the vectors $(2, 0, -3)$ and $(5, 1, 0)$ are not comparable with respect to this order. Thus, the law of trichotomy does not hold in $\mathbb{R}^n$ with respect to this order for $n > 1$.

It is now interesting to note that most of the properties of $\delta$ on $\mathbb{R}$ hold for the extended $\delta$ on $\mathbb{R}^n$.

**2.3 Proposition:** $\delta$ in $\mathbb{R}^n$ satisfies the following:

(a) $\delta(x) \ge 0$ for all $x \epsilon \mathbb{R}^n$ and $\delta(x) = 0$ if and only if$x = 0$.

(b) $\delta(rx) = \delta(x)$ for all $x \epsilon \mathbb{R}^n$ and for all $0 \ne r \epsilon \mathbb{R}$.

(c) $\delta(x + y) \le \delta(x) + \delta(y)$for all $x, y \epsilon \mathbb{R}^n$.

**Proof:**

Let $X = (x_1, x_2, \dots, x_n)$ *and* $Y = (y_1, y_2, \dots, y_n) \epsilon \mathbb{R}^n$

Then, clearly $\delta\,(x) = (\delta\,(x_1), \delta\,(x_2), \dots, \delta\,(x_n)) \ge 0$.

since $\delta\,(x_i) \ge 0$ for all $i$.

And $\delta\,(x) = 0$ if and only if $\delta\,(x_i) = 0$ for all $i$.

if and only if $x_i = 0$ for all $i$ if and only if $x = 0$.

For $r \ne 0$, consider

$$\delta\,(rx) = (\delta\, rx_1, rx_2, \dots, rx_n)$$

$$\delta\,(x) = (\delta\,(rx_1), \delta\,(rx_2), \dots, \delta\,(rx_n))$$

$$\delta\,(x) = (\delta\,(x_1), \delta\,(x_2), \dots, \delta\,(x_n)) = \delta\,(x)$$

Further,

$$\delta\,(x + y) = \delta\,(x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$$

$$= (\delta\,(x_1 + y_1), \delta\,(x_2 + y_2), \dots, \delta\,(x_n + y_n)) \qquad (9)$$

Now, $\delta(x_i + y_i) \le \delta(x_i) + \delta(y_i)$ for all $i$.

So, (9) gives

$$\delta(x + y) = \big(\delta\,(x_1) + \delta\,(y_1)\big), \Big(\delta\,(x_2) + \delta\,(y_2), \dots, \big(\delta\,(x_n) + \delta\,(n)\big)\Big),$$

$$= (\delta\,(x_1), \delta\,(x_2), \dots \delta\,(x_n)) + \big(\delta\,(y), \delta\,(y_2), \dots, \delta\,(y_n)\big)$$

For an $X = (x_1, x_2, \dots, x_n)$in $\mathbb{R}^n$, its norm $\|x\|$ is defined by

$\|x\| = (x_1{}^2 + x_2{}^2 + \cdots + x_n{}^2)^{1/2}$. (10) For

$x = (x_1, x_2, \dots, x_n)$, put $|x| = (|x_1|, |x_2|, \dots, |x_n|)$. (11)

Also, for $r \epsilon \mathbb{R}^n, put\ \bar{r} = (r, r, .., r) \epsilon \mathbb{R}^n$ . (12)

For example, $\bar{1} = (1, 1, .., 1)$.

Now we have the result:

**2.4 Proposition**

Let $x \epsilon \mathbb{R}^n$.

(a) If $x \geq 1$, then $\|x\| \geq \|\delta(x)\|$

(b) If $|x| \leq 1$, then $\|x\| \leq \|\delta(x)\|$.

**Proof:**

Suppose $x \geq 1$. That is, $x_i \geq 1$ for all $i$.

So, $x_i{}^2 \geq 1^2 = \delta(x_i)^2$for all $i$.

Hence, $x_1{}^2 + x_2{}^2 +, \dots, x_n{}^2 \geq \delta(x_1)^2 + \delta(x_2)^2 + \cdots + \delta(x_n)^2$,

which implies

$$\|x\| \geq \|\delta(x)\|$$

Now, if $|x| \leq 1$, then $|x_i| \leq 1$, for all $i$.

so that $x_i{}^2 \leq 1^2 = \delta(x_i)^2$

Hence we get, $\|x\| \leq \|\delta(x)\|$

**Note:**

It is not true that $x \leq 1$ implies $\|x\| \leq \|\delta(x)\|$

For example, let $X = (-2, 0, 0) in\ \mathbb{R}^3$

Then $x \leq 1$.

$\delta(x) = (1, 0, 0)$ and $\|\delta(x)\| = 1$.

But $\|x\| = 2 \geq \|\delta(x)\|$.

**2.5 Definition:**

For $i = 1, 2, \dots, n$ , $e_i$ is the vector defined by

$e_i = (0,0, \dots, 1, 0, , \dots, 0)$, with 1 occurs in the $i^{th}$place and all other co ordinators are 0.

**Remark:**

$$\delta(e_i) = e_i$$

More generally, for $r \neq 0$,

$$\delta(re_i) = \delta\,(0,0,\dots,r,0,0,\dots,0)$$

$$= (0,0,\dots,\delta(r),0,0,\dots,0)$$

$$= (0,0,\dots,1,0,0,\dots,0) = e_i$$

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