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Research Article

Description Of 2 Local Bilateral Symmetric Multiplications Of Functional Component Matrix In Banach Algebra

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Abstract. This article covers studying the description of 2 local bilateral symmetric multiplications in Banach algebra of functional component matrix. The definitions related to this are mentioned, and the lemma and theorem are proven.

Keywords: two-dimensional matrix, identity matrix, Banach algebra, 2 local bilateral symmetric multiplication, linear operator, associative algebra.

Definition 1. Let us see two-dimensional matrix algebra as $M_2(R)$. Let us say that if to reflect $\Delta: M_2(R) \to M_2(R)$ we take $x, y \in M_2(R)$, and there is such $A \in M_2(R)$, in this case if $\Delta(x) = AXA, \Delta(y) = AYA$ equality is fulfilled, Δ is called two local bilateral multiplication.

Theorem 1. Let us mark the continued functions complex of two dimensional matrix algebra as $M_2(R) \otimes C[a,b]$.

Let there be $\Delta: M_2(R) \otimes C[a,b] \to M_2(R) \otimes C[a,b]$ as such two local bilateral multiplications, for the matrix $a = \begin{pmatrix} a_{3.1}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{pmatrix}$ to satisfy the condition $x, y \epsilon M_2(R) \otimes C[a,b]$, $\Delta(x) = axa$ $\Delta(y) = aya$, $f(t) > 0, g(t) > 0, k(t) > 0, h(t) > 0$ in a random interval of $x \epsilon [a,b]$.

Then Δ is bilateral multiplication which means such $a \epsilon M_2(R) \otimes C[a,b]$ is defined and for a random $x \epsilon M_2(R) \otimes C[a,b]$, $\Delta(x) = axa$ is appropriate.

Proving. For a random $x \epsilon M_2(R) \otimes C[a,b]$

$$\Delta(x) = BxB\ ,\ \ \Delta(e_{3.1}(t)) = Be_{3.1}(t)B$$
$$\Delta(x) = CxC\ ,\ \ \Delta(e_{12}(t)) = Ce_{12}(t)C$$
$$\Delta(x) = DxD\ ,\ \ \Delta(e_{21}(t)) = De_{21}(t)D$$
$$\Delta(x) = FxF\ ,\ \ \Delta(e_{22}(t)) = Fe_{22}(t)F$$

According to the lemma, there is such A

$$Be_{3.1}(t)B = Ae_{3.1}(t)A, \qquad Ce_{12}(t)C = Ae_{12}(t)A,$$
$$De_{21}(t)D = Ae_{21}(t)A, \ \ Fe_{22}(t)F = Ae_{22}(t)A$$

the equality is appropriate, which means, we can get

$$\begin{cases} b_{3.1}^2(t) = a_{3.1}^2(t) \\ b_{3.1}(t)b_{12}(t) = a_{3.1}(t)a_{12}(t) \\ b_{3.1}(t)b_{21}(t) = a_{3.1}(t)a_{21}(t) \\ b_{21}(t)b_{12}(t) = a_{21}(t)a_{12}(t) \end{cases} \qquad \begin{cases} c_{3.1}(t)c_{21}(t) = a_{3.1}(t)a_{21}(t) \\ c_{3.1}(t)c_{22}(t) = a_{3.1}(t)a_{22}(t) \\ c_{21}^2(t) = a_{21}^2(t) \\ c_{21}(t)c_{22}(t) = a_{21}(t)a_{22}(t) \end{cases}$$

$$\begin{cases} d_{3.1}(t)d_{12}(t) = a_{3.1}(t)a_{12}(t) \\ d_{12}^2(t) = a_{12}^2(t) \\ d_{3.1}(t)d_{22}(t) = a_{3.1}(t)a_{22}(t) \\ d_{22}(t)d_{12}(t) = a_{22}(t)a_{12}(t) \end{cases} \qquad \begin{cases} f_{21}(t)f_{12}(t) = a_{21}(t)a_{12}(t) \\ f_{12}(t)f_{22}(t) = a_{12}(t)a_{22}(t) \\ f_{21}(t)f_{22}(t) = a_{21}(t)a_{22}(t) \\ f_{22}^2(t) = a_{22}^2(t) \end{cases}$$

From this,

$$a_{3.1}(t) = b_{3.1}(t) = c_{3.1}(t) = d_{3.1}(t), \qquad a_{12}(t) = b_{12}(t) = f_{12}(t) = d_{12}(t),$$
$$a_{21}(t) = b_{21}(t) = c_{21}(t) = f_{21}(t), \qquad a_{22}(t) = c_{22}(t) = d_{22}(t) = f_{22}(t)$$

In order to make it comfortable, we enter the following markings,

$$\alpha_{3.1}(t) = a_{3.1}^2(t)x_{3.1} + a_{3.1}(t)a_{12}(t)x_{21} + a_{3.1}(t)a_{21}(t)x_{12} + a_{12}(t)a_{21}(t)x_{22},$$
$$\alpha_{12} = a_{3.1}(t)a_{12}(t)x_{3.1} + a_{12}^2(t)x_{21} + a_{3.1}(t)a_{22}(t)x_{12} + a_{12}(t)a_{22}(t)x_{22},$$
$$\alpha_{21}(t) = a_{21}(t)a_{3.1}(t)x_{3.1} + a_{3.1}(t)a_{22}(t)x_{21} + a_{21}^2(t)x_{12} + a_{21}(t)a_{22}(t)x_{22},$$
$$\alpha_{22}(t) = a_{21}(t)a_{12}(t)x_{3.1} + a_{12}(t)a_{22}(t)x_{21} + a_{3.1}(t)a_{22}(t)x_{21} + a_{22}^2(t)x_{22},$$
$$\beta_{3.1}(t) = b_{3.1}^2(t)x_{3.1} + b_{3.1}(t)b_{12}(t)x_{21} + b_{3.1}(t)b_{21}(t)x_{12} + b_{12}(t)b_{21}(t)x_{22},$$
$$\gamma_{21}(t) = c_{21}(t)c_{3.1}(t)x_{3.1} + c_{3.1}(t)c_{22}(t)x_{21} + c_{21}^2(t)x_{12} + c_{21}(t)c_{22}(t)x_{22},$$
$$\delta_{12}(t) = d_{3.1}(t)d_{12}(t)x_{3.1} + d_{12}^2(t)x_{21} + d_{3.1}(t)d_{22}(t)x_{12} + d_{12}(t)d_{22}(t)x_{22},$$
$$\varepsilon_{22}(t) = f_{21}(t)f_{12}(t)x_{3.1} + f_{12}(t)f_{22}(t)x_{21} + f_{3.1}(t)f_{22}(t)x_{21} + f_{22}^2(t)x_{22}.$$

Then,

$$\Delta(x) = BXB = \begin{pmatrix} \beta_{3.1}(t) & \alpha_{12}(t) \\ \alpha_{21}(t) & \alpha_{22}(t) \end{pmatrix}, \qquad \Delta(x) = CXC = \begin{pmatrix} \alpha_{3.1}(t) & \alpha_{12}(t) \\ \gamma_{21}(t) & \alpha_{22}(t) \end{pmatrix},$$
$$\Delta(x) = DXD = \begin{pmatrix} \alpha_{3.1}(t) & \delta_{12}(t) \\ \alpha_{21}(t) & \alpha_{22}(t) \end{pmatrix}, \qquad \Delta(x) = FXF = \begin{pmatrix} \alpha_{3.1}(t) & \alpha_{12}(t) \\ \alpha_{21}(t) & \varepsilon_{22}(t) \end{pmatrix}.$$

It means,

$$\Delta(x) = BXB = CXC = DXD = FXF = \begin{pmatrix} \alpha_{3.1}(t) & \alpha_{12}(t) \\ \alpha_{21}(t) & \varepsilon_{22}(t) \end{pmatrix}.$$

The theorem is proven.

Definition 2. If there is such $a \epsilon M_2(\mathbb{C})$ symmetric matrix, the equalities as $\varphi(x) = axa$, $\varphi(y) = aya$ are fulfilled for a random $x, y \epsilon M_2(\mathbb{C})$, φ is called 2 local symmetric bilateral multiplication.

Theorem 2. The random 2 local linear reflection of a limited dimensional vector space is a linear operator on this vector space.

Theorem 3. $M_n(\mathbb{C})$ can be a random 2 local symmetric bilateral multiplication linear operator in associative algebra.

Proving. As we know, if $\varphi(x) = bxb$, $x\epsilon M_2(\mathbb{C})$ and $b\epsilon M_2(\mathbb{C})$, then a φ is linear operator and is defined by any d.

Let us assume, there is a random 2 local symmetric bilateral reflection defined on n =2 and $\varphi - M_2(\mathbb{C})$. Then, for the random $x\epsilon M_2(\mathbb{C})$ such $a\epsilon M_2(\mathbb{C})$ is defined,

$\varphi(x) = axa$:

$$\varphi(x) = \varphi\begin{pmatrix} x_{11} \\ x_{21} \\ x_{12} \\ x_{22} \end{pmatrix} = \begin{pmatrix} a_{11}a_{12} \\ a_{21}a_{22} \end{pmatrix}\begin{pmatrix} x_{11}x_{12} \\ x_{21}x_{22} \end{pmatrix}\begin{pmatrix} a_{11}a_{12} \\ a_{21}a_{22} \end{pmatrix} =$$

$$= \begin{pmatrix} a_{11}x_{11} + a_{12}x_{21}a_{11}x_{12} + a_{12}x_{22} \\ a_{21}x_{11} + a_{22}x_{21}a_{21}x_{12} + a_{22}x_{22} \end{pmatrix}\begin{pmatrix} a_{11}a_{12} \\ a_{21}a_{22} \end{pmatrix}$$

For the comfort

$$\propto_{11} = a_{11}x_{11}a_{11} + a_{12}x_{21}a_{11} + a_{11}x_{12}a_{21} + a_{12}x_{22}a_{21} ,$$

$$\propto_{12} = a_{11}x_{11}a_{12} + a_{12}x_{21}a_{12} + a_{11}x_{12}a_{22} + a_{12}x_{22}a_{22},$$

$$\propto_{21} = a_{21}x_{11}a_{11} + a_{22}x_{21}a_{11} + a_{21}x_{12}a_{21} + a_{22}x_{22}a_{21} ,$$

$$\propto_{22} = a_{21}x_{11}a_{12} + a_{22}x_{21}a_{12} + a_{21}x_{12}a_{22} + a_{22}x_{22}a_{22}$$

if to enter marking as this way, we get

$$\varphi(x) = \begin{pmatrix} \propto_{11}\propto_{12} \\ \propto_{21}\propto_{22} \end{pmatrix} = \begin{pmatrix} a_{11}^2 & a_{12}a_{11} & a_{11}a_{21} & a_{12}a_{21} \\ a_{21}a_{11} & a_{22}a_{11} & a_{21}^2 & a_{22}a_{21} \\ a_{11}a_{12} & a_{12}^2 & a_{11}a_{22} & a_{12}a_{22} \\ a_{21}a_{12} & a_{22}a_{12} & a_{21}a_{22} & a_{22}^2 \end{pmatrix}\begin{pmatrix} x_{11} \\ x_{21} \\ x_{12} \\ x_{22} \end{pmatrix}$$

If matrix a is symmetric, this formulation can be formed in the following way:

$$\begin{pmatrix} a_{11}^2 & a_{11}a_{12} & a_{11}a_{12} & a_{12}^2 \\ a_{11}a_{12} & a_{11}a_{22} & a_{21}^2 & a_{22}a_{12} \\ a_{11}a_{12} & a_{12}^2 & a_{11}a_{22} & a_{12}a_{22} \\ a_{12}^2 & a_{22}a_{12} & a_{12}a_{22} & a_{22}^2 \end{pmatrix}\begin{pmatrix} x_{11} \\ x_{21} \\ x_{12} \\ x_{22} \end{pmatrix}.$$

Here we can see that here the first multiplied matrix is symmetric. Similar $\varphi(y) = aya$ is also defined through the above mentioned matrix. According to the above mentioned theorem 1, φ is a linear operator.

Let us say that there is a random 2 local symmetric bilateral reflection which is defined on n =3 and $\varphi - M_2(\mathbb{C})$. Then, for the random $x\epsilon M_2(\mathbb{C})$, such $a\epsilon M_2(\mathbb{C})$ is defined that $(x) = axa$:

$$\varphi(x) = \varphi \begin{pmatrix} x_{11} \\ x_{21} \\ x_{31} \\ x_{12} \\ x_{11} \\ x_{22} \\ x_{32} \\ x_{13} \\ x_{23} \\ x_{33} \end{pmatrix} = \begin{pmatrix} a_{11} a_{12} a_{13} \\ a_{21} a_{22} a_{23} \\ a_{31} a_{32} a_{33} \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{13} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} \begin{pmatrix} a_{11} a_{12} a_{13} \\ a_{21} a_{22} a_{23} \\ a_{31} a_{32} a_{33} \end{pmatrix}$$

Let us enter markings in the following way:

$$\gamma_{11} = \sum_{m=1}^{3} \left(\left(\sum_{k=1}^{3} a_{1k} x_{km} \right) a_{m1} \right), \quad \gamma_{12} = \sum_{m=1}^{3} \left(\left(\sum_{k=1}^{3} a_{1k} x_{km} \right) a_{m2} \right),$$

$$\gamma_{13} = \sum_{m=1}^{3} \left(\left(\sum_{k=1}^{3} a_{1k} x_{km} \right) a_{m1} \right), \quad \gamma_{21} = \sum_{m=1}^{3} \left(\left(\sum_{k=1}^{3} a_{2k} x_{km} \right) a_{m1} \right),$$

$$\gamma_{22} = \sum_{m=1}^{3} \left(\left(\sum_{k=1}^{3} a_{2k} x_{km} \right) a_{m2} \right), \quad \gamma_{23} = \sum_{m=1}^{3} \left(\left(\sum_{k=1}^{3} a_{2k} x_{km} \right) a_{m3} \right),$$

$$\gamma_{31} = \sum_{m=1}^{3} \left(\left(\sum_{k=1}^{3} a_{3k} x_{km} \right) a_{m1} \right), \quad \gamma_{32} = \sum_{m=1}^{3} \left(\left(\sum_{k=1}^{3} a_{3k} x_{km} \right) a_{m2} \right),$$

$$\gamma_{33} = \sum_{m=1}^{3} \left(\left(\sum_{k=1}^{3} a_{3k} x_{km} \right) a_{m3} \right).$$

Then,

$$\varphi(x) = \begin{pmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{pmatrix}$$

The proving is completed.

3. Let us assume that on the complex figures $M_n(C)C$ there is matrix algebra with measure n. If on taking a random matrix $x, y \in M_n(C)$ for reflecting $\Delta: M_n(C) \to M_n(\mathrm{C})$, there is such $A \in M_n(C)$, in this case the equalities $\Delta(x) = AxA$, $\Delta(y) = AyA$ are fulfilled, then Δ is called 2 local bilateral multiplication.

According to this entered notion, the following lemma is appropriate. Let us assume that Δ is $M_n(R)$ 2 local bilateral multiplication which is positive in two-dimensional matrix algebra.

Lemma 1 In n measured matrix algebra, there is such $A \in M_n(R)$ that for the identity matrix $e_{ij} \in M_n(R), i,j = 1,2,\dots,n$ on Δ two local bilateral multiplication the equality $\Delta(e_{ij}) = Ae_{ij}A$ is fulfilled, which means:

$$\Delta(e_{ij}) = Ae_{ij}A, i,j = 1,2,\dots,n$$

Proving.

$$\Delta(e_{ij}) = A(ij,kl)e_{ij}A(ij,kl) = (a_{ij,kl}^{\alpha i} \cdot a_{ij,kl}^{j\beta})_{\alpha,\beta=1}^{n},$$

$(ij) \neq (kl), \quad \mathrm{i},j,k,l = 1,2,\dots,n$

Here

$A(ij,kl) = A(kl,ij)$

Let us calculate for each matrix the two local bilateral multiplication for identity matrixes. It is appropriate if the following equalities are $i,j = 1,2,\dots,n$:

$$A(ij,11)e_{ij}A(ij,11) = A(ij,12)e_{ij}A(ij,12) =$$
$$= A(ij,13)e_{ij}A(ij,13) = \cdots = A(ij,nn)e_{ij}A(ij,nn)$$

Here, it is clear that $A(ij,ij)e_{ij}A(ij,ij)$ is skipped.

Theorem 4. Let us say F is a random field, and Δ is two local bilateral multiplication which is $A = \{a_{ij}, a_{ij} > 0\}$ and all the components of which are positive in two-dimensional matrix algebras $M_n(F)$, which means it is two local bilateral multiplication defined by matrixes. Then, there is such $A \in M_n(F)$ that in this case for a random $X \in M_n(\mathrm{F})$ the equality $\Delta(X) = AXA$ is fulfilled, which means Δ is bilateral multiplication.

Proving. Let us take a random element $x \epsilon M_n(R)$. Let us assume that the following equalities are fulfilled in such $\{B(ij)\}_{i,j=1}^{n}$ matrix system

$$\Delta(x) = B(ij)xB(ij)$$
$$\Delta(e_{i,j}) = B(i,j)e_{i,j}B(i,j), \quad i,j = \overline{1,n}.$$

According to the lemma 1, there is such A that the equalities

$$\Delta(x) = B(ij)e_{ij}B(ij) = Ae_{ij}A, \quad i,j = \overline{1,n}$$

are appropriate. Let us equalize all the components in the multiplications of the equality:

$$b_{i,j}^{\alpha,i} \cdot b_{i,j}^{j,\beta} = a^{\alpha i} \cdot a^{j\beta}, \quad \alpha,\beta = \overline{1,n}, \ i,j = \overline{1,n}.$$

We get the following equality system from this equality:

$$\begin{cases} b_{ij}^{\alpha i} = a^{\alpha i} & \alpha = \overline{1,n} \\ b_{ij}^{j\beta} = a^{j\beta} & \beta = \overline{1,n} \end{cases}.$$

This system equalizes the components like i — column, j — row.

Let us use this equality system for the following matrix equalities.

$\Delta(x) = B(ij)xB(ij) i,j = \overline{1,n}$. Then, according to the above mentioned equality

$$\Delta(x) = B(ij)xB(ij) = \left(\sum_{k=1}^{n} b_{ij}^{\alpha k}\, x^{k\beta}\right)_{\alpha,\beta=1}^{n}.$$

$$B(ij) = \left(\sum_{l=1}^{n}\left(\sum_{k=1}^{n} b_{ij}^{\alpha k}\, x^{kl}\right) b_{ij}^{l\beta}\right)_{\alpha,\beta=1}^{n}$$

If $\alpha = j$, and $\beta = i$

$$\sum_{l=1}^{n}\left(\sum_{k=1}^{n} b_{ij}^{jk}\, x^{kl}\right) b_{ij}^{li} = \sum_{l=1}^{n}\left(\sum_{k=1}^{n} a^{jk} x^{kl}\right) a^{li} = (AxA)_{ji}.$$

The components j — row and i — column of each $B(ij)xB(ij)$ multiplication matrix are equal to $(AxA)_{ji}$, here $i,j = \overline{1,n}$

$$\Delta(x) = B(11)xB(11) = B(12)xB(12) = \cdots = B(nn)xB(nn)$$

The equalities produce the equality $\Delta(x) = AxA$.

Consequently, there is such $A \in M_n(C)$ that if to take a random $x \in M_n(C)$, the equality $\Delta(X) = AXA$ is appropriate.

The theorem is proven.

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