J.R. Alieva
Turkish Online Journal of Qualitative Inquiry (TOJQI)
Volume 12, Issue6, July , 7111- 7116

Research Article

Description Of 2 Local Bilateral Symmetric Multiplications Of Functional Component Matrix In Banach Algebra

J.R. Alieva (Phd, ADU, Zhralieva@Mail.Ru)

Abstract. This article covers studying the description of 2 local bilateral symmetric multiplications in Banach algebra of functional component matrix. The definitions related to this are mentioned, and the lemma and theorem are proven.

Keywords: two-dimensional matrix, identity matrix, Banach algebra, 2 local bilateral symmetric multiplication, linear operator, associative algebra.

Definition 1. Let us see two-dimensional matrix algebra as $M_2(R)$. Let us say that if to reflect $\Delta: M_2(R) \to M_2(R)$ we take $x, y \in M_2(R)$, and there is such $A \in M_2(R)$, in this case if $\Delta(x) = AXA, \Delta(y) = AYA$ equality is fulfilled, Δ is called two local bilateral multiplication.

Theorem 1. Let us mark the continued functions complex of two dimensional matrix algebra as $M_2(R) \otimes C[a,b]$.

Let there be $\Delta: M_2(R) \otimes C[a,b] \to M_2(R) \otimes C[a,b]$ as such two local bilateral multiplications, for the matrix $a = \begin{pmatrix} a_{3.1}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{pmatrix}$ to satisfy the condition $x, y \epsilon M_2(R) \otimes C[a,b]$, $\Delta(x) = axa$ $\Delta(y) = aya$, $f(\mathrm{t}) > 0, g(t) > 0, k(t) > 0, h(t) > 0$ in a random interval of $x\epsilon[a,b]$.

Then Δ is bilateral multiplication which means such $a\epsilon M_2(R) \otimes C[a,b]$ is defined and for a random $x\epsilon\, M_2(R) \otimes C[a,b], \Delta(x) = axa$ is appropriate.

Proving. For a random $x\epsilon M_2(R) \otimes C[a,b]$

$$\Delta(x) = BxB\,,\ \ \Delta(e_{3.1}(t)) = Be_{3.1}(t)B$$
$$\Delta(x) = CxC\,,\ \ \Delta(e_{12}(t)) = Ce_{12}(t)C$$
$$\Delta(x) = DxD\,,\ \ \Delta(e_{21}(t)) = De_{21}(t)D$$
$$\Delta(x) = FxF\,,\ \ \Delta(e_{22}(t)) = Fe_{22}(t)F$$

According to the lemma, there is such A

$$Be_{3.1}(t)B = Ae_{3.1}(t)A, \qquad Ce_{12}(t)C = Ae_{12}(t)A,$$
$$De_{21}(t)D = Ae_{21}(t)A,\ \ Fe_{22}(t)F = Ae_{22}(t)A$$

the equality is appropriate, which means, we can get

$$\begin{cases} b_{3.1}^2(t) = a_{3.1}^2(t) \\ b_{3.1}(t)b_{12}(t) = a_{3.1}(t)a_{12}(t) \\ b_{3.1}(t)b_{21}(t) = a_{3.1}(t)a_{21}(t) \\ b_{21}(t)b_{12}(t) = a_{21}(t)a_{12}(t) \end{cases} \qquad \begin{cases} c_{3.1}(t)c_{21}(t) = a_{3.1}(t)a_{21}(t) \\ c_{3.1}(t)c_{22}(t) = a_{3.1}(t)a_{22}(t) \\ c_{21}^2(t) = a_{21}^2(t) \\ c_{21}(t)c_{22}(t) = a_{21}(t)a_{22}(t) \end{cases}$$

$$\begin{cases} d_{3.1}(t)d_{12}(t) = a_{3.1}(t)a_{12}(t) \\ d_{12}^2(t) = a_{12}^2(t) \\ d_{3.1}(t)d_{22}(t) = a_{3.1}(t)a_{22}(t) \\ d_{22}(t)d_{12}(t) = a_{22}(t)a_{12}(t) \end{cases} \qquad \begin{cases} f_{21}(t)f_{12}(t) = a_{21}(t)a_{12}(t) \\ f_{12}(t)f_{22}(t) = a_{12}(t)a_{22}(t) \\ f_{21}(t)f_{22}(t) = a_{21}(t)a_{22}(t) \\ f_{22}^2(t) = a_{22}^2(t) \end{cases}$$

From this,

$$a_{3.1}(t) = b_{3.1}(t) = c_{3.1}(t) = d_{3.1}(t), \qquad a_{12}(t) = b_{12}(t) = f_{12}(t) = d_{12}(t),$$

$$a_{21}(t) = b_{21}(t) = c_{21}(t) = f_{21}(t), \qquad a_{22}(t) = c_{22}(t) = d_{22}(t) = f_{22}(t)$$

In order to make it comfortable, we enter the following markings,

$$\alpha_{3.1}(t) = a_{3.1}^2(t)x_{3.1} + a_{3.1}(t)a_{12}(t)x_{21} + a_{3.1}(t)a_{21}(t)x_{12} + a_{12}(t)a_{21}(t)x_{22},$$

$$\alpha_{12} = a_{3.1}(t)a_{12}(t)x_{3.1} + a_{12}^2(t)x_{21} + a_{3.1}(t)a_{22}(t)x_{12} + a_{12}(t)a_{22}(t)x_{22},$$

$$\alpha_{21}(t) = a_{21}(t)a_{3.1}(t)x_{3.1} + a_{3.1}(t)a_{22}(t)x_{21} + a_{21}^2(t)x_{12} + a_{21}(t)a_{22}(t)x_{22},$$

$$\alpha_{22}(t) = a_{21}(t)a_{12}(t)x_{3.1} + a_{12}(t)a_{22}(t)x_{21} + a_{3.1}(t)a_{22}(t)x_{21} + a_{22}^2(t)x_{22},$$

$$\beta_{3.1}(t) = b_{3.1}^2(t)x_{3.1} + b_{3.1}(t)b_{12}(t)x_{21} + b_{3.1}(t)b_{21}(t)x_{12} + b_{12}(t)b_{21}(t)x_{22},$$

$$\gamma_{21}(t) = c_{21}(t)c_{3.1}(t)x_{3.1} + c_{3.1}(t)c_{22}(t)x_{21} + c_{21}^2(t)x_{12} + c_{21}(t)c_{22}(t)x_{22},$$

$$\delta_{12}(t) = d_{3.1}(t)d_{12}(t)x_{3.1} + d_{12}^2(t)x_{21} + d_{3.1}(t)d_{22}(t)x_{12} + d_{12}(t)d_{22}(t)x_{22},$$

$$\varepsilon_{22}(t) = f_{21}(t)f_{12}(t)x_{3.1} + f_{12}(t)f_{22}(t)x_{21} + f_{3.1}(t)f_{22}(t)x_{21} + f_{22}^2(t)x_{22}.$$

Then,

$$\Delta(x) = BXB = \begin{pmatrix} \beta_{3.1}(t) & \alpha_{12}(t) \\ \alpha_{21}(t) & \alpha_{22}(t) \end{pmatrix}, \qquad \Delta(x) = CXC = \begin{pmatrix} \alpha_{3.1}(t) & \alpha_{12}(t) \\ \gamma_{21}(t) & \alpha_{22}(t) \end{pmatrix},$$

$$\Delta(x) = DXD = \begin{pmatrix} \alpha_{3.1}(t) & \delta_{12}(t) \\ \alpha_{21}(t) & \alpha_{22}(t) \end{pmatrix}, \qquad \Delta(x) = FXF = \begin{pmatrix} \alpha_{3.1}(t) & \alpha_{12}(t) \\ \alpha_{21}(t) & \varepsilon_{22}(t) \end{pmatrix}.$$

It means,

$$\Delta(x) = BXB = CXC = DXD = FXF = \begin{pmatrix} \alpha_{3.1}(t) & \alpha_{12}(t) \\ \alpha_{21}(t) & \varepsilon_{22}(t) \end{pmatrix}.$$

The theorem is proven.

Definition 2. If there is such $a \epsilon M_2(\mathbb{C})$ symmetric matrix, the equalities as $\varphi(x) = axa$, $\varphi(y) = aya$ are fulfilled for a random $x, y \epsilon M_2(\mathbb{C})$, φ is called 2 local symmetric bilateral multiplication.

Theorem 2. The random 2 local linear reflection of a limited dimensional vector space is a linear operator on this vector space.

Theorem 3. $M_n(\mathbb{C})$ can be a random 2 local symmetric bilateral multiplication linear operator in associative algebra.

Proving. As we know, if $\varphi(x) = bxb$, $x\epsilon M_2(\mathbb{C})$ and $b\epsilon M_2(\mathbb{C})$, then a φ is linear operator and is defined by any d.

Let us assume, there is a random 2 local symmetric bilateral reflection defined on n =2 and $\varphi - M_2(\mathbb{C})$. Then, for the random $x\epsilon M_2(\mathbb{C})$ such $a\epsilon M_2(\mathbb{C})$ is defined,

$\varphi(x) = axa$:

$$\varphi(x) = \varphi\begin{pmatrix} x_{11} \\ x_{21} \\ x_{12} \\ x_{22} \end{pmatrix} = \begin{pmatrix} a_{11}a_{12} \\ a_{21}a_{22} \end{pmatrix}\begin{pmatrix} x_{11}x_{12} \\ x_{21}x_{22} \end{pmatrix}\begin{pmatrix} a_{11}a_{12} \\ a_{21}a_{22} \end{pmatrix} =$$

$$= \begin{pmatrix} a_{11}x_{11} + a_{12}x_{21}a_{11}x_{12} + a_{12}x_{22} \\ a_{21}x_{11} + a_{22}x_{21}a_{21}x_{12} + a_{22}x_{22} \end{pmatrix}\begin{pmatrix} a_{11}a_{12} \\ a_{21}a_{22} \end{pmatrix}$$

For the comfort

$$\propto_{11} = a_{11}x_{11}a_{11} + a_{12}x_{21}a_{11} + a_{11}x_{12}a_{21} + a_{12}x_{22}a_{21} ,$$

$$\propto_{12} = a_{11}x_{11}a_{12} + a_{12}x_{21}a_{12} + a_{11}x_{12}a_{22} + a_{12}x_{22}a_{22},$$

$$\propto_{21} = a_{21}x_{11}a_{11} + a_{22}x_{21}a_{11} + a_{21}x_{12}a_{21} + a_{22}x_{22}a_{21} ,$$

$$\propto_{22} = a_{21}x_{11}a_{12} + a_{22}x_{21}a_{12} + a_{21}x_{12}a_{22} + a_{22}x_{22}a_{22}$$

if to enter marking as this way, we get

$$\varphi(x) = \begin{pmatrix} \propto_{11}\propto_{12} \\ \propto_{21}\propto_{22} \end{pmatrix} = \begin{pmatrix} a_{11}^2 & a_{12}a_{11} & a_{11}a_{21} & a_{12}a_{21} \\ a_{21}a_{11} & a_{22}a_{11} & a_{21}^2 & a_{22}a_{21} \\ a_{11}a_{12} & a_{12}^2 & a_{11}a_{22} & a_{12}a_{22} \\ a_{21}a_{12} & a_{22}a_{12} & a_{21}a_{22} & a_{22}^2 \end{pmatrix}\begin{pmatrix} x_{11} \\ x_{21} \\ x_{12} \\ x_{22} \end{pmatrix}$$

If matrix a is symmetric, this formulation can be formed in the following way:

$$\begin{pmatrix} a_{11}^2 & a_{11}a_{12} & a_{11}a_{12} & a_{12}^2 \\ a_{11}a_{12} & a_{11}a_{22} & a_{21}^2 & a_{22}a_{12} \\ a_{11}a_{12} & a_{12}^2 & a_{11}a_{22} & a_{12}a_{22} \\ a_{12}^2 & a_{22}a_{12} & a_{12}a_{22} & a_{22}^2 \end{pmatrix}\begin{pmatrix} x_{11} \\ x_{21} \\ x_{12} \\ x_{22} \end{pmatrix}.$$

Here we can see that here the first multiplied matrix is symmetric. Similar $\varphi(y) = aya$ is also defined through the above mentioned matrix. According to the above mentioned theorem 1, φ is a linear operator.

Let us say that there is a random 2 local symmetric bilateral reflection which is defined on n =3 and $\varphi - M_2(\mathbb{C})$. Then, for the random $x\epsilon M_2(\mathbb{C})$, such $a\epsilon M_2(\mathbb{C})$ is defined that $(x) = axa$:

$$\varphi(x) = \varphi\begin{pmatrix} x_{11} \\ x_{21} \\ x_{31} \\ x_{12} \\ x_{11} \\ x_{22} \\ x_{32} \\ x_{13} \\ x_{23} \\ x_{33} \end{pmatrix} = \begin{pmatrix} a_{11}a_{12}a_{13} \\ a_{21}a_{22}a_{23} \\ a_{31}a_{32}a_{33} \end{pmatrix}\begin{pmatrix} x_{11}\ x_{12}\ x_{13} \\ x_{21}x_{22}\ x_{13} \\ x_{31}x_{32}\ x_{33} \end{pmatrix}\begin{pmatrix} a_{11}a_{12}a_{13} \\ a_{21}a_{22}a_{23} \\ a_{31}a_{32}a_{33} \end{pmatrix}$$

Let us enter markings in the following way:

$$\gamma_{11} = \sum_{m=1}^{3}\left(\left(\sum_{k=1}^{3} a_{1k}x_{km}\right)a_{m1}\right), \quad \gamma_{12} = \sum_{m=1}^{3}\left(\left(\sum_{k=1}^{3} a_{1k}x_{km}\right)a_{m2}\right),$$

$$\gamma_{13} = \sum_{m=1}^{3}\left(\left(\sum_{k=1}^{3} a_{1k}x_{km}\right)a_{m1}\right), \quad \gamma_{21} = \sum_{m=1}^{3}\left(\left(\sum_{k=1}^{3} a_{2k}x_{km}\right)a_{m1}\right),$$

$$\gamma_{22} = \sum_{m=1}^{3}\left(\left(\sum_{k=1}^{3} a_{2k}x_{km}\right)a_{m2}\right), \quad \gamma_{23} = \sum_{m=1}^{3}\left(\left(\sum_{k=1}^{3} a_{2k}x_{km}\right)a_{m3}\right),$$

$$\gamma_{31} = \sum_{m=1}^{3}\left(\left(\sum_{k=1}^{3} a_{3k}x_{km}\right)a_{m1}\right), \quad \gamma_{32} = \sum_{m=1}^{3}\left(\left(\sum_{k=1}^{3} a_{3k}x_{km}\right)a_{m2}\right),$$

$$\gamma_{33} = \sum_{m=1}^{3}\left(\left(\sum_{k=1}^{3} a_{3k}x_{km}\right)a_{m3}\right).$$

Then,

$$\varphi(x) = \begin{pmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{pmatrix}$$

The proving is completed.

3. Let us assume that on the complex figures $M_n(C)C$ there is matrix algebra with measure n. If on taking a random matrix $x\,, y \in\ M_n(C)$ for reflecting $\Delta: M_n(C) \to M_n(\mathrm{C})$, there is such $A \in M_n(C)$, in this case the equalities $\Delta(x) = AxA$, $\Delta(y)\ = AyA$ are fulfilled, then Δ is called 2 local bilateral multiplication.

According to this entered notion, the following lemma is appropriate. Let us assume that Δ is $M_n(R)$ 2 local bilateral multiplication which is positive in two-dimensional matrix algebra.

Lemma 1 In n measured matrix algebra, there is such $A \in M_n(R)$ that for the identity matrix $e_{ij} \in M_n(R), i,j = 1,2,\dots,n$ on Δ two local bilateral multiplication the equality $\Delta(e_{ij}) = Ae_{ij}A$ is fulfilled, which means:

$$\Delta(e_{ij}) = Ae_{ij}A, i,j = 1,2,\dots,n$$

Proving.

$$\Delta(e_{ij}) = A(ij,kl)e_{ij}A(ij,kl) = (a_{ij,kl}^{\alpha i} \cdot a_{ij,kl}^{j\beta})_{\alpha,\beta=1}^{n},$$

$(ij) \neq (kl)$, $i,j,k,l = 1,2,\dots,n$

Here

$A(ij,kl) = A(kl,ij)$

Let us calculate for each matrix the two local bilateral multiplication for identity matrixes. It is appropriate if the following equalities are $i,j = 1,2,\dots,n$:

$$A(ij,11)e_{ij}A(ij,11) = A(ij,12)e_{ij}A(ij,12) =$$
$$= A(ij,13)e_{ij}A(ij,13) = \dots = A(ij,nn)e_{ij}A(ij,nn)$$

Here, it is clear that $A(ij,ij)e_{ij}A(ij,ij)$ is skipped.

Theorem 4. Let us say F is a random field, and Δ is two local bilateral multiplication which is $A = \{a_{ij}, a_{ij} > 0\}$ and all the components of which are positive in two-dimensional matrix algebras $M_n(F)$, which means it is two local bilateral multiplication defined by matrixes. Then, there is such $A \in M_n(F)$ that in this case for a random $X \in M_n(\mathrm{F})$ the equality $\Delta(X) = AXA$ is fulfilled, which means Δ is bilateral multiplication.

Proving. Let us take a random element $x \epsilon M_n(R)$. Let us assume that the following equalities are fulfilled in such $\{B(ij)\}_{i,j=1}^{n}$ matrix system

$$\Delta(x) = B(ij)xB(ij)$$
$$\Delta(e_{i,j}) = B(i,j)e_{i,j}B(i,j), \quad i,j = \overline{1,n}.$$

According to the lemma 1, there is such A that the equalities

$$\Delta(x) = B(ij)e_{ij}B(ij) = Ae_{ij}A, \quad i,j = \overline{1,n}$$

are appropriate. Let us equalize all the components in the multiplications of the equality:

$$b_{i,j}^{\alpha,i} \cdot b_{i,j}^{j,\beta} = a^{\alpha i} \cdot a^{j\beta}, \quad \alpha,\beta = \overline{1,n}, \; i,j = \overline{1,n}.$$

We get the following equality system from this equality:

$$\begin{cases} b_{ij}^{\alpha i} = a^{\alpha i} & \alpha = \overline{1,n} \\ b_{ij}^{j\beta} = a^{j\beta} & \beta = \overline{1,n} \end{cases}.$$

This system equalizes the components like i – column, j – row.

Let us use this equality system for the following matrix equalities.

$\Delta(x) = B(ij)xB(ij) i,j = \overline{1,n}$. Then, according to the above mentioned equality

$$\Delta(x) = B(ij)xB(ij) = \left(\sum_{k=1}^{n} b_{ij}^{\alpha k}\, x^{k\beta}\right)_{\alpha,\beta=1}^{n}.$$

$$B(ij) = \left(\sum_{l=1}^{n}\left(\sum_{k=1}^{n} b_{ij}^{\alpha k}\, x^{kl}\right) b_{ij}^{l\beta}\right)_{\alpha,\beta=1}^{n}$$

If $\alpha = j$, and $\beta = i$

$$\sum_{l=1}^{n}\left(\sum_{k=1}^{n} b_{ij}^{jk}\, x^{kl}\right) b_{ij}^{li} = \sum_{l=1}^{n}\left(\sum_{k=1}^{n} a^{jk} x^{kl}\right) a^{li} = (AxA)_{ji}.$$

The components j – row and i – column of each $B(ij)xB(ij)$ multiplication matrix are equal to $(AxA)_{ji}$, here $i,j = \overline{1,n}$

$$\Delta(x) = B(11)xB(11) = B(12)xB(12) = \cdots = B(nn)xB(nn)$$

The equalities produce the equality $\Delta(x) = AxA$.

Consequently, there is such $A \in M_n(C)$ that if to take a random $x \in M_n(C)$, the equality $\Delta(X) = AXA$ is appropriate.

The theorem is proven.

REFERENCES:

1. Albeverio S., Ayupov Sh.A., Abdullayev R.Z. Arens spaces associated with von Neumann Algebras and Normal Spaces, SFB 611, Universitat Bonn, Preprint, No 381, 2008.
2. Dales H.G., Banach algebras and automatic continuity. Clarendon Press 2000.
3. Rudin U. Functional Analysis. – M.: Mir, 1975.
4. Sarimsakov T.A. The course of functional analysis – Tashkent, O`qituvchi, 1980.
5. Segal I. A non-commutative extension of abstract integration, Ann. Math. 57 (1953) 401-457.
6. Sherstenev A.N., Lugovaya G.D. Functional Analysis. – Kazan, 2008.
7. Feintuch A., Saeks R. *System theory. A. Hilbert space approach* .-New York, London: Academic Press, 1982. p. 310.
8. Murphy D. C^*– algebras and operator theory, M. Faktorial, 1997.
9. Sadovnichiy V.A. Operator Theory. – M.: Vysshaya shokla, 1999.
10. Iosida K. Functional Analysis. –M.: Mir, 1967.