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On Classes Of Defensive Alliance Difference Secure Sets Of A Graph

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Abstract

For a graph $G = (V, E)$, a defensive alliance of G is a set of vertices $S \subseteq V(G)$ satisfying the condition that for each $v \in S$, at least half of the vertices in the closed neighborhood of v are in S. Let $\phi: V(G) \to \{1, 2, 3, \dots, |V|\}$ be a bijection. A subset $S \subseteq V$ is called difference secure set of G with respect to ϕ if for all $u, v \in S$, there is a $w \in S$ such that $|\phi(u) - \phi(v)| = \phi(w)$ if and only if $uv \in E$. A defensive alliance S of G which is also a difference secure set is called defensive alliance difference secure set. In this paper, we compute the maximum cardinality of various types of minimal defensive alliance difference secure sets for paths.

Keywords: defensive alliance, difference labeling, difference secure sets, defensive alliance difference secure sets.

Introduction

For a simple, connected graph $G = (V, E)$, S is a non-empty subset of $V(G)$, $\langle S \rangle$ denotes the subgraph of G induced by S and compliment of S is $\overline{S} = V - S$. Let p_1 be the property of the graph satisfied by at least one subset S among the varieties of $V(G)$ subset. Then such subsets satisfying p_1-property will have 4 different types of sets and defined as shown in Table 1.

Table 1: Varieties of sets with p_1-property.

Sets	Condition for the set S	Condition for the set $\overline{S}$
p_1-set	S should satisfy the p_1-property	No condition
$p_1{}^*$-set	S should satisfy the p_1-property.	$\overline{S}$ should satisfy the p_1-property.
P_1-set	S should satisfy the p_1-property.	$\overline{S}$ should not satisfy the p_1-property.
$P_1{}^*$-set	S should not satisfy the p_1-property.	$\overline{S}$ should not satisfy the p_1-property.

Let p_2 be one more property satisfied by any subset of varieties of subsets of $V(G)$. If any subset has to satisfy both p_1 and p_2 property then we get 4^2 varieties of p_1p_2-sets of G. With the properties p_1, p_2 we get various types of p_1p_2-sets of G. In general, with the properties p_1, p_2, p_3,..., p_k we get 4^k different types of $p_1p_2p_3 \dots p_k$-sets of G. These properties are studied for subsets of 2^n of the $V(G)$ of the order n. By removing null set and whole set from the subsets of $V(G)$, we make the analysis mo re relevant. Hence, for $2^n - 2$ subsets of $V(G)$, we review the properties. A p_1p_2-set is said to be a mi nimal p_1p_2-set of G if none of its proper subsets are p_1p_2-set of G. The maximum cardinality of a min imal p_1p_2-set of G is called upper p_1p_2 number and is denoted by $u_{p_1p_2}(G)$.

For a vertex v in a graph $G = (V, E)$, the open neighbourhood of v is the set $N(v) = \{u: uv \in E\}$ a nd the closed neighbourhood of v is $N[v] = N(v) \cup \{v\}$. A non-empty set of vertices $S \subseteq V$ is called d efensive alliance if for every $v \in S$, $|N[v] \cap S| \geq |N(v) \cap \overline{S}|$. In this case, we say that every vertex in S is defended from possible attack by the vertices in $\overline{S}$. The initial studies on defensive alliance is done by S. M. Hedetniemi, S.T. Hedetniemi and P. Kristiansen (2004). Also, a very useful survey is done b y H. Fernau and J. A. Rodriguez-Velazquez (2014) on alliance and related parameters in graphs.

We recall Theorem 1.1 and Theorem 1.2 studied by P. Kristiansen, S. M. Hedetniemi and S. T. He detniemi (2002) for immediate reference.

Theorem 1.1: The subgraph induced by a minimal defensive alliance set of a connected graph G is co nnected.

Theorem 1.2: For any graph G, $a(G) = 1$ if and only if there exists a vertex $v \in V(G)$ such that $\deg(v) \leq 1$.

Remark 1.3: For a path P_n, any vertex $v_i \in V$ with $\deg(v_i) = 2$, the set $S = \{v_i\}$ is not a defensive all iance.

A graph G is called difference graph if there exists a mapping labeling ϕ between the vertex set of G into distinct positive integers S so that the edges in G exists if and only if the difference of its end vertices is the label of a vertex in G. Further research work on difference graph is found in [4-6]. A subset $S \subseteq V(G)$ is called a difference secure set of G with respect to $\phi: V \longrightarrow \{1, 2, 3, \dots, n\}$ if for all $u, v \in S$, there is a $w \in S$ such that $|\phi(u) - \phi(v)| = \phi(w)$ if and only if $uv \in E$. Among all such ϕ the maximum cardinality of a difference secure set is called difference secure number of G and it is denoted by $DSN(G)$.

B. Sooryanarayana and Suma A.S (2018) studied neighborhood resolving property of a graph G. Similarly, for any graph G, we obtain 16 varieties of subsets $S \subset V(G)$ which have both defensive alliance property and a difference secure property as shown in Table 2.

Table 2: Varieties of sets with a-property and d-property

ad-set	ad*-set	aD-set	aD*-set.
a*d-set	a*d*-set	a*D-set	a*D*-set
Ad-set	Ad*-set	AD-set	AD*-set
A*d-set	A*d-set	A*D-set	A*D*-set

The number of defensible members who can defend immediately in alliances is determined by the codes assigned to them. If the members of the alliances are neighbors and those members will always be able to protect themselves without delay in time, the disparity in the codes assigned to them also exists. If not, there are no codes, and members do not protect them.

Remark 1.4: For any graph $G = (V, E)$, the singleton set $S = \{v\}$, $v \in V$ is always difference secure set.

Remark 1.5: For a connected graph G with order $n \geq 2$, the subset $S \subset V(G)$ with $|S| = 2$ is always a difference secure set.

Remark 1.6: For any $a > 0$, the difference secure set S for any triangle free graph G ($n \geq 3$), cannot contain a subset with labels $\{a, 2a, 3a\}$.

Remark 1.7: For any path P_n with difference secure set S, if $d \in \phi(S)$ then the set $\{x, x+d, x+2d\} \not\subseteq \phi(S)$, for any $x > d$.

Theorem 1.8 and Theorem 1.9 is referred from Sunita Priya D'Silva. (2020) research work.

Theorem 1.8: For a path P_n of order n, $DSN(P_n) = \left\lceil \frac{n}{2} \right\rceil + 1$.

Theorem 1.9: For $n \geq 11$, the graph P_n cannot have both sets S and $\overline{S}$ as difference secure simultaneously.

Observation 1.10: For a graph P_n, $2 \leq n \leq 10$, there exists a set S for which both S and $\overline{S}$ are difference secure.

Lemma 1.11: For any path P_n of order n, D^*-set does not exist.

Results and discussion

In this sections, the maximum cardinality of ad-set for a path P_n are discussed.

Lemma 2.1: For a path P_n, any subset S of $V(P_n)$, if $\langle S \rangle$ and $\langle \overline{S} \rangle$ of P_n are connected then S cannot be a A-set.

Theorem 2.2: For any path P_n, $u_{ad}(P_n) = u_{a^*d}(P_n) = \begin{cases} 1, & \text{for } n = 2, 3 \\ 2, & \text{for } n \geq 4. \end{cases}$

Proof. For any path P_n the singleton set S with pendant vertex is a minimal ad-set. For $n = 2, 3$, maximum cardinality of minimal set is one. Hence, $u_{ad}(P_n) = u_{a^*d}(P_n) = 1$. When $n \geq 4$, consider $S = \{v_i, v_{i+1}\}$, $2 \leq i \leq n-2$. Then the sets S and $\overline{S}$ both are defensive alliance. Also, the subsets of S, $\{v_i\}$ or $\{v_{i+1}\}$ are not defensive alliance (since they are not end vertices). Hence, S is a minimal defensive set. Any a-set or a^*-set w cardinality three of P_n will contain the set of the form S. Hence, maximum cardinality of minimal a-set and a^*-set is 2. We define labeling function $\phi: V(G) \to \{1, 2, 3, \ldots, n\}$ as follows: $\phi(v_i) = a$ and $\phi(v_{i+1}) = 2a$ for $a \in Z^+$. Since v_i and v_{i+1} are adjacent vertices in S and $|\phi(v_{i+1}) - \phi(v_i)| = a = \phi(v_i)$ implies $v_i \in S$ satisfying the condition of d-set. Therefore, the set S will be a ad-set and a^*d-set. Hence, $u_{ad}(P_n) = u_{a^*d}(P_n) = 2$.

Theorem 2.3: For any path P_n, $u_{Ad}(P_n) = \begin{cases} \text{does not exist} & \text{for } n = 2 \\ 2 & \text{for } n = 3 \\ 3 & \text{for } n \geq 4. \end{cases}$

Proof. If v is a pendant vertex of the path P_n, then both the subsets $S = \{v\}$ and $\overline{S} = V - \{v\}$ will be a defensive alliance. It is obvious that any subset S of $V(P_2)$ does not satisfy condition of A-set. Further, if v is not a pendant vertex then the set S itself is not a defensive alliance. Hence, S to be an A-set, we must have $|S| \geq 2$. Also, by Remark 1.5, S is a d-set. Therefore, $u_{Ad}(P_3) = 2$.

For the set $S = \{v_i, v_j\}$, $1 \leq j \leq n$, with v_i as end vertex of P_n and if $v_j \sim v_i$, then both S and $\overline{S}$ will be defensive alliance. If not, then S itself fails to be a defensive alliance. Hence, we take $|S| > 2$. Let $S = \{v_i, v_j, v_k\}$, $1 \leq j < k \leq n$, with v_i as end vertex. If S contains all the three non-adjacent vertices, then S is not a defensive alliance. Hence, we choose a set S with $v_i \nsim v_j$ and $v_j \sim v_k$. For this S we define a labeling function $\phi: V \to \{1, 2, 3, \ldots, n\}$ as follows. $\phi(v_j) = 2\phi(v_k)$ with $\phi(v_k) = a \in \left\{1, 2, 3, \ldots \left\lfloor \frac{n}{2} \right\rfloor\right\}$ and $\phi(v_i) \not\equiv 0 \pmod a$. Hence, S is a Ad-set. Also, there exist no minimal set S with $|S| > 3$ which is a Ad-set. Therefore, $u_{Ad}(P_n) = 3$.

Theorem 2.4: For any path P_n, $u_{Ad^*}(P_n) = \begin{cases} 2, & \text{for } n = 3 \\ 3, & \text{for } 4 \leq n \leq 8 \\ \text{does not exist}, & \text{for } n = 2 \text{ and } n \geq 9. \end{cases}$

Proof. Let $S \subset V(P_n)$. Any singleton set S is not a A-set. Clearly, P_2 has no set which is a A-set. For P_3, $S = \{v_1, v_3\}$ is a A-set (since $\overline{S}$ is a singleton set). For $n \geq 4$, the set $S = \{v_1, v_3, v_4\}$ or $S = \{v_{n-3}, v_{n-2}, v_n\}$ is a A-set (as discussed in Theorem 2.3). Now if this set S has to be a d^*-set then there should exit some labeling function ϕ such that both S and $\overline{S}$ are difference secure. We define a labeling function $\phi: V(P_n) \to \{1, 2, 3, \ldots, n\}$ as follows:

For $n = 3$, $\phi(S) = \{1, 3\}$ and $\phi(\bar{S}) = \{2\}$. Here, S is also a d^*-set. Hence, $u_{Ad^*}(P_3) = 2$. Consider the set $S = \{v_1, v_3, v_4\}$ with label $\phi(v_3) = 2\phi(v_4)$ and the pendant vertex v_1 can be chosen such that $\phi(v_1) \neq 2\phi(v_3)$ and $\phi(v_1) \neq 2\phi(v_4)$. Then S is a difference secure. Similarly, the set $\{v_{n-3}, v_{n-2}, v_n\}$ is also a difference secure. For the above set S and for each $4 \leq n \leq 8$, label the following subsets as $\phi(\bar{S}) = \{1\}$, $\phi(\bar{S}) = \{1, 5\}$, $\phi(\bar{S}) = \{5, 1, 2\}$, $\phi(\bar{S}) = \{5, 3, 7, 4\}$ and $\phi(\bar{S}) = \{5, 6, 3, 7, 4\}$. We observe that the above sets $\bar{S}$ are difference secure. Hence, S is a d^*-set. Also, we see that there is no other minimal Ad^*-set S, with $|S| > 3$. Therefore, $u_{Ad^*}(P_n) = 3$.

Consider the set $S = \{v_2, v_3, v_4, v_5\}$ with $\phi(S) = \{2, 1, 9, 8\}$ for $n = 9$. Then we get $\bar{S} = \{v_1, v_6, v_7, v_8, v_9\}$ with $\phi(\bar{S}) = \{5, 4, 7, 3, 6\}$. Similarly, for $n = 10$, consider $S = \{v_1, v_2, v_3, v_4, v_5\}$ with $\phi(S) = \{5, 10, 9, 1, 2\}$ and $\bar{S} = \{v_6, v_7, v_8, v_9, v_{10}\}$ with $\phi(\bar{S}) = \{6, 3, 7, 4, 8\}$. This set S is a unique d^*-set but not a A-set (Since both S and $\bar{S}$ are defensive alliance). Also, from Theorem 1.9, d^*-set does not exist for $n \geq 11$. Hence, for $n \geq 9$, $u_{Ad^*}(P_n)$ does not exist.

Theorem 2.5: For any path P_n, $u_{A^*d}(P_n) = \begin{cases} \text{does not exist}, & \text{for } n = 2, 3. \\ 2, & \text{for } n \geq 4. \end{cases}$

Proof. Let S be a subset of $V(P_n)$. We know that singleton set $S = \{v\}$, where v is a pendant vertex, is always a defensive alliance. Also, if v is not a pendant vertex then $\bar{S}$ becomes defensive alliance. Hence, S is not an A^*-set. Therefore, we must have $|S| \geq 2$. Let $S = \{v_{i-1}, v_{i+1}\}$ for $2 \leq i \leq n-1$ contains one pair of non-adjacent vertices. Then obviously, $\bar{S}$ will also contain at least one pair of non-adjacent vertices. Hence from Remark 1.3, S is an A^*-set. Let $\phi: V \to \{1, 2, 3, \ldots, n\}$ be a labeling function. Suppose we label $\phi(v_{i-1}) = 1$ and $\phi(v_{i+1}) = 3$ then S will be difference secure set (Since $|\phi(v_{i-1}) - \phi(v_{i+1})| = 2 \notin S$). Hence, S is both defensive alliance and difference secure set. Also, there is no minimal A^*-set with $|S| > 2$. Therefore, $u_{A^*d}(P_n) = 2$.

Theorem 2.6: For any path P_n, $u_{A^*D}(P_n) = \begin{cases} \text{does not exist}, & \text{for } n = 2, 3 \\ 2, & \text{for } n \geq 4. \end{cases}$

Proof. The set $S = \{v_i, v_{i+2}\}$, $1 \leq i \leq n-2$, is a minimal A^*-set (Since S and $\bar{S}$ are not a defensive alliance). For $n = 2, 3$, we cannot have such A^*-set. Therefore, $u_{A^*D}(P_2)$ and $u_{A^*D}(P_3)$ does not exist.

Let us consider a labeling function $\phi: V \to \{1, 2, 3, \ldots, n\}$. We have to show that there exist atleast one difference secure set S, for which $\bar{S}$ is not a difference secure set. From Remark 1.5, the set $S = \{v_2, v_4\}$ is difference secure. For $n \geq 4$, $\bar{S}$ is not a difference secure. Since, when $n = 4, 5$, we label $\bar{S}$ as $\phi(\bar{S}) = \{2, 4\}$, $\phi(\bar{S}) = \{2, 4, 5\}$ respectively, which is clearly not a difference secure. When $n = 6, 7$, there exist no labeling for $\bar{S}$ which is difference secure and for $n \geq 8$, $S = \{v_2, v_4\}$, we get $|\bar{S}| > \left\lceil \frac{n}{2} \right\rceil + 1$ which implies from Theorem 1.8, $\bar{S}$ is not a difference secure set. Hence, $u_{A^*D}(P_n) = 2$.

Theorem 2.7. For any path P_n, $u_{A^*d^*}(P_n) = \begin{cases} 2, & \text{for } n = 4, 5. \\ 3, & \text{for } n = 6,7,8. \\ \text{does not exist}, & \text{for } n = 2,3 \text{ and } n \geq 9. \end{cases}$

Proof. Let $\phi : V \to \{1, 2, 3, \ldots, n\}$ be a labeling function and $S \subset V(P_n)$. If the sets S and $\bar{S}$ will contain atleast a vertex other than pendant vertex having no neighbor vertices in S and $\bar{S}$ respectively, then S will be a A^*-set. From Remark 1.3, in P_2 and P_3 there exists no such subset S. Therefore, $u_{A^*d^*}(P_2)$ and $u_{A^*d^*}(P_3)$ does not exist.

Define the labeling ϕ for the set S (note that S is the set of darkened vertices in the below figures) which is a A^*-set. For $n = 4, 5$, the labeling ϕ is shown in Figure 1. Hence, $u_{A^*d^*}(P_n) = u_{A^*d^*}(P_n) = 2$.

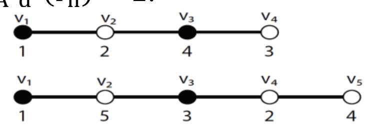

Figure 1: Labeling of P_4 and P_5

From Theorem 1.8, for $n > 6$, if $|S| = 2$ then, $\overline{S}$ is not a difference secure set. When $n = 6$, for the set $\{v_i, v_{i+1}\}$, $1 \leq i \leq n-2$, $\overline{S}$ is not a difference secure set. Hence, we take $|S| > 2$. For $6 \leq n \leq 8$, the labeling ϕ is shown in Figure 2. Therefore, $u_{A^*d^*}(P_6) = u_{A^*d^*}(P_7) = u_{A^*d^*}(P_8) = 3$.

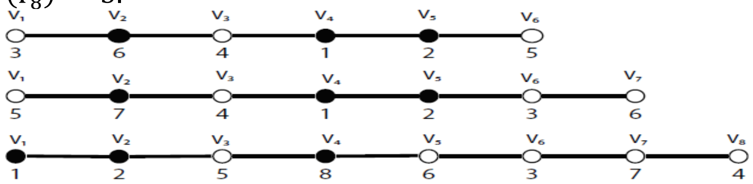

Figure 2: Labeling of P_6, P_7 and P_8

When $n = 9, 10$, there exist unique set S which is d^*-set but fails to be A^*-set. The labeling of the only set S which is d^*-set is shown in Figure 3.

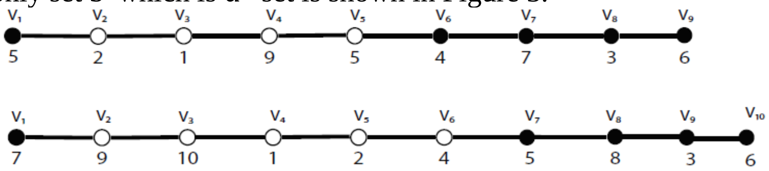

Figure 3: Labeling of P_9 and P_{10}

Also, by Theorem 1.9, for $n \geq 11$, d^*-set does not exist. Hence, $n \geq 9$, $u_{A^*d^*}(P_n)$ does not exist.

Theorem 2.8. For any path P_n, $u_{ad^*}(P_n) = u_{a^*d^*}(P_n) = \begin{cases} 1, & \text{for } n = 2,3. \\ 2, & \text{for } n = 4,7. \\ 3, & \text{for } n = 5,6,8. \\ 4, & \text{for } n = 9. \\ 5, & \text{for } n = 10. \\ \text{does not exist}, & \text{for } n \geq 11. \end{cases}$

Proof. For P_2 and P_3, the singleton set $S = \{v\}$ (where v is a pendent vertex) is a minimal a-set and a^*-set with maximal cardinality. From Remark 1.4, S is also a d^*-set. Hence, $u_{ad^*}(P_n) = u_{a^*d^*}(P_n) = 1$. For any path P_n, $n \geq 4$, the singleton set $S = \{v\}$ is minimal set not having maximum cardinality. Clearly the set $S = \{v_i, v_{i+1}\}, 2 \leq i \leq n-2$ is a minimal set (since subsets $\{v_i\} \subset S$ or $\{v_{i+1}\} \subset S$ are not defensive alliance). But $u_{ad^*}(P_n)$ and $u_{a^*d^*}(P_n)$ will vary according to the difference secure property of P_n. Consider the labeling function $\phi: V(P_n) \to \{1, 2, 3, \dots, n\}$ for each P_n, $n \geq 4$ as follows:
(i) For P_4, $S = \{v_2, v_3\}$ and $\overline{S} = \{v_1, v_4\}$ with $\phi(v_2) = 1$, $\phi(v_3) = 2$, $\phi(v_1) = 3$ and $\phi(v_4) = 4$. Therefore, $u_{ad^*}(P_4) = u_{a^*d^*}(P_4) = 2$.
(ii) For P_5, $S = \{v_2, v_3\}$, we can have $\phi(S) = \{1, 2\}$ or $\phi(S) = \{2, 4\}$. But for both the possibilities, $\overline{S} = \{v_1, v_4, v_5\}$ will never be difference secure for any ϕ. Hence, $|S| > 2$. Therefore, we take $= \{v_2, v_3, v_4\}$ with $\phi(S) = \{1, 2, 4\}$ and $\phi(\overline{S}) = \{3, 5\}$.
(iii) For P_6, $S = \{v_2, v_3, v_4\}$ with $\phi(S) = \{1, 2, 4\}$ and $\phi(\overline{S}) = \{5, 3, 6\}$. For P_8, $S = \{v_3, v_4, v_5\}$ with $\phi(S) = \{5, 7, 2\}$ and $\phi(\overline{S}) = \{3, 6, 4, 8\}$. Hence $u_{ad^*}(P_n) = u_{a^*d^*}(P_n) = 3$ for $n = 5, 6, 8$.
(iv) But for P_7, consider $S = \{v_2, v_3\}$, with $\phi(v_2) = 1$, $\phi(v_3) = 2$ and $\overline{S} = \{v_1, v_4, v_5, v_6, v_7\}$ with $\phi(v_1) = 5$, $\phi(v_4) = 4$, $\phi(v_5) = 7, \phi(v_6) = 3$ and $\phi(v_7) = 6$. Hence, $u_{ad^*}(P_7) = u_{a^*d^*}(P_7) = 2$.
(v) For P_9, consider $S = \{v_2, v_3, v_4, v_5\}$ with $\phi(S) = \{2, 1, 9, 8\}$ and $\overline{S} = \{v_1, v_6, v_7, v_8, v_9\}$ with $\phi(\overline{S}) = \{5, 4, 7, 3, 6\}$. Therefore, $u_{ad^*}(P_9) = u_{a^*d^*}(P_9) = 4$.
(vi) For P_{10}, consider $S = \{v_1, v_2, v_3, v_4, v_5\}$ with $\phi(S) = \{5, 10, 9, 1, 2\}$ and $\overline{S} = \{v_6, v_7, v_8, v_9, v_{10}\}$ with $\phi(\overline{S}) = \{6, 3, 7, 4, 8\}$. Hence, $u_{ad^*}(P_{10}) = u_{a^*d^*}(P_{10}) = 5$.
(vii) For $n \geq 11$, from Theorem 1.9, d^*-set does not exist.

Theorem 2.9. For any path P_n,

$$u_{aD}(P_n) = u_{a^*D}(P_n) = \begin{cases} \text{does not exist,} & \text{for } n \leq 4. \\ 1, & \text{for } n = 7. \\ 2, & \text{for } n = 5, 6, n \geq 8. \end{cases}$$

Proof. The singleton set $S = \{v_1\}$ or $S = \{v_n\}$ is defensive alliance for a path P_n with minimal set. From Remark 1.4, a singleton set S is a difference secure set. It is obvious that, P_2 does not have D-set. The set $S_1 = \{v_i, v_{i+1}\}$ for $2 \leq i \leq n-2$ is a minimal defensive alliance, since no proper subset of it is a defensive alliance. Clearly $\overline{S_1}$ is also defensive alliance. Hence, S_1 is a-set and a*-set with maximum cardinality. Let $\phi : V \to \{1, 2, 3, \ldots, n\}$ be a labeling function. From Remark 1.5, the set S_1 is difference secure with $\phi(v_i) = 2\phi(v_{i+1})$. When $n = 3, 4$, from Remark 1.4, the set $\overline{S_1}$ is also difference secure. Hence, $u_{aD}(P_n) = u_{a^*D}(P_n)$ does not exist.

For $n = 5$, we are left with labels for $\overline{S_1}$ as $\phi(\overline{S_1}) = \{3, 4, 5\}$ or $\phi(\overline{S_1}) = \{1, 3, 5\}$. We observe that $\overline{S_1} = \{v_1, v_4, v_5\}$ is not a difference secure set. Similarly, when $n = 6$, we get $\phi(\overline{S_1}) = \{3, 4, 5, 6\}$ or $\phi(\overline{S_1}) = \{1, 3, 5, 6\}$ or $\phi(\overline{S_1}) = \{1, 2, 4, 5\}$. Clearly, $\overline{S_1} = \{v_1, v_4, v_5, v_6\}$ is not a difference secure set. For $n \geq 8$, we get $|\overline{S_1}| \geq \left\lceil \frac{n}{2} \right\rceil + 1$. From Theorem 1.8, $\overline{S_1}$ is not a difference secure set. Hence, for $n = 5, 6$ and $n \geq 8$ we get $u_{aD}(P_n) = u_{a^*D}(P_n) = 2$. But for $n = 7$, there exist a set S_1 with $\phi(S_1) = \{1, 2\}$ and $\phi(\overline{S_1}) = \{5, 4, 7, 3, 6\}$. Then $\overline{S_1}$ is difference secure set. Therefore, the set S_1 is not a D-set. For any singleton set S, $|\overline{S}| = 6 > \left\lceil \frac{n}{2} \right\rceil + 1$. Hence, $\overline{S}$ is not a difference secure set. Therefore, $u_{aD}(P_7) = u_{a^*D}(P_7) = 1$.

Theorem 2.10. For any path P_n, $u_{AD}(P_n) = \begin{cases} \text{does not exist,} & \text{for } 2 \leq n \leq 5 \\ 3, & \text{for } n = 6, 7 \\ 4, & \text{for } n \geq 8 \end{cases}$

Proof. Let $\phi : V(P_n) \to \{1, 2, 3, \ldots, n\}$ be the labeling function. Obviously, for $n = 2$, the set S is not a A-set. For $n = 3$, the set $S = \{v_1, v_3\}$ is a A-set. But by Remark 1.4 we get $|\overline{S}| = 1$ making $\overline{S}$ as a difference secure set. Therefore, S is not a D-set. When $n = 4, 5$, the set $S = \{v_1, v_3, v_4\}$ is a minimal A-set with maximum cardinality. Then, $|\overline{S}| = 1$, which is a difference secure set. Hence, S is not a D-set. Therefore, $u_{AD}(P_n)$ does not exist for $2 \leq n \leq 5$.

For $n \geq 6$, if we take $S = \{v_i, v_{i+1}, v_{i+3}, v_{i+4}\}$, $2 \leq i \leq n-5$ which is also a minimal A-set with maximum cardinality (as no proper subset of S is A-set). This set S contains two pair of adjacent vertices. But for $n = 6, 7$, it is not possible to label these four vertices v_2, v_3, v_5, v_6 by any labeling function ϕ. Hence, $|S| \leq 3$. Therefore, for $n = 6, 7$, take $S = \{v_1, v_3, v_4\}$ with $\phi(S) = \{6, 1, 2\}$ and $\phi(S) = \{5, 1, 2\}$ respectively. This set S will be both defensive alliance and difference secure. Hence, $u_{AD}(P_n) = 3$. When $n \geq 8$, we consider the set $S = \{v_i, v_{i+1}, v_{i+3}, v_{i+4}\}$. We label $\phi(v_i) = 2\phi(v_{i+1})$ and $\phi(v_{i+2}) = 2\phi(v_{i+3})$. Then from Remark 1.7, $\overline{S}$ is not difference secure set. Hence, $u_{AD}(P_n) = 4$.

The following Theorem is obvious from the Lemma 1.11.

Theorem 2.11. For every integer $n \geq 3$, $u_{aD^*}(P_n) = u_{a^*D^*}(P_n) = u_{AD^*}(P_n) = u_{A^*D^*}(P_n)$ does not exist.

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References

1. Badekara Sooryanarayana, Silvia. L. Sequeira, M. Vishu Kumar (2019), On the neighborhood alliance sets in a graph, J. Adv. Res. Dyn. Control Syst. 11(10), 66–74.
2. Badekara. Sooryanarayana and Suma A.S (2018). On classes of neighborhood resolving sets of a graph. Electron. J. Graph Theory Appl. (EJGTA), **6**(1), 29-36.
3. Badekara Sooryanarayana and Sunita Priya D’Silva (2020), Mod difference labeling of some classes of digraphs Malaya Journal of Matematik, **8**(1), 32-36.
4. Bloom G. S, P.Hell and H.Taylor (1979). Collecting autographs: n-node that have n -integers.

Annals of New York Academy of Sciences, 93–102.

5. Chandru Hegde, B. Sooryanarayana (2019), Strong alliances in graphs, Commun. Comb. Optim. 4(1), 1-13.
6. Chandru. Hegde, B. Sooryanarayana, Silvia. L. Sequeira and M. Vishu Kumar (2020), Accurate alliances in graphs, J. Adv. Res. Dyn. Control Syst. 12(4) (2020), 203–215.
7. Fernau H and J. A. Rodriguez-Velazquez (2014). A survey on alliances and related parameters in graphs. Electron. J. Graph Theory Appl. (EJGTA), **2**(1), 70–86.
8. Hedetniemi S. M., S. T. Hedetniemi and P. Kristiansen (2004). Alliances in graphs. J. Combin. Math. Combin.Comput., **48,** 157–177.
9. Joseph A. Gallian(2019) . A dynamic survey of graph labeling. The Electron. J. Com., #DS6.
10. Kristiansen P, S. M. Hedetniemi and S. T. Hedetniemi (2002). Introduction to alliances in graphs. (IBID) Proc. 17th Internat. Symp. Comput. Infom. Sci. ISCIS , **17**, 308–312.
11. Silvia L. Sequeira, B. Sooryanarayana and Chandru hedge (2021), Neighborhood alliance in join of a graph with K_1, J. Math. Comput. Sci. 11 (2021), No. 3, 2624-2649.
12. Sunita Priya D'Silva.(2020), Difference labelling of graphs, PhD Thesis, Visvesvaraya Technological University, Belgaum, Karnataka, India.