Turkish Online Journal of Qualitative Inquiry (TOJQI)
Volume 12, Issue 6, July, 2021: 9150 - 9158

Research Article

On the Convergence and Stability of AP Iteration Process in CAT(0) Spaces

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Abstract
In this paper, we use the AP iteration method in the context of CAT(0) space to approximate invariant points for contraction maps. We also prove that our iteration process for contraction map is faster than the leading Picard-S iteration process to demonstrate the proposed algorithm's convergence behaviour. To support the analytic proofs, numerical examples are given. Furthermore, we demonstarte that the AP iteration method is T-stable.
2010 AMS Subject Classification: 47H10, 54H25, 54E50.
Keywords: CAT(0) space, Iteration process, Stability.

1 Introduction
Fixed point theory consumes a lot of literature on the subject because it provides useful methods for resolving a wide range of problems in fields as an example architecture, chemistry, economics and game theory, among others. It is difficult to find the value of a fixed point for a map until it has been verified, so we use iterative procedures to measure it. Many iteration methods have emerged over time, making it impossible to cover them all. Picard iteration procedure is used to approach the fixed point in the well-known Banach contraction theorem. Mann [14], Ishikawa [12], Agarwal [2] , Noor [15], Abbas [1], SP [16], CR [7], Normal-S [19], Picard-S [9], and Thakur New [20] are some of the other known iteration procedures.
Fastness and stability are critical factors in whether or not an iteration process is chosen over another iteration process. Rhoades [17] stated in 1991 that the Mann iteration process converges faster than the Ishikawa iteration process for decreasing function, but the Ishikawa iteration process is stronger for increasing function. It also appears that the Mann iteration process is unaffected by the initial guess (see Rhoades [18]). The authors of Agarwal et al. [2] believed that the Agarwal iteration method for contraction maps converges at the same pace as Picard iteration procedure and is faster than Mann iteration procedure. The authors argue in Abbas and Nazir [1] that their iteration method converges faster than the Agarwal iteration process. Also, Chugh et al. [7] looked at the strong convergence of CR iteration process for quasi-contractive operators. Meanwhile, according to Gursoy and Karakaya [9], Picard-S iteration method converges faster than all the above mentioned iteration procedures for contraction maps. Thakur et al. [20], showed that the Thakur new iteration method converges faster than many existing iteration procedures in literature for the maps which pleases condition(C) using a numerical example.
Many other stability outcomes for numerous invariant point iterative algorithms and for different groups of nonlinear maps were established based on the findings of Harder [10], Harder and Hicks [11], who introduced and studied the definition of stable fixed point iterative algorithm. Following

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their results, many authors studied the stability outcomes for the three step iteration process (see [4, 21, 22]).

In light of the above, we present the AP iteration method in the context of CAT(0) space and demonstarte in detail that it is stable. Then we show it converges faster for contraction maps than Picard-S, Thakur new iteration processes . In the current literature of contraction maps, we equate the convergence of the AP iteration method with Picard-S and Thakur new iteration processes numerically.

2 Preliminaries

“There is a systematic discussion of CAT(0) spaces and their importance in different branches of mathematics are given [5, 8]. For the sake of simplicity, we recall a few definitions and conclusions.

Lemma 2.1 *[6] Let E be a CAT(0) space. Then*

$$d((1-s)x\oplus sy,z)\leq(1-s)d(x,z)+sd(y,z) \text{ for all } x,y,z\in E \text{ and } s\in[0,1].$$

Definition 2.2 *[3] Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be two real convergent sequences with limits a and b, respectively. Then we say that $\{a_n\}_{n=0}^{\infty}$ converge faster than $\{b_n\}_{n=0}^{\infty}$ if $lim_{n\to\infty}\frac{d(a_n,a)}{d(b_n,b)}=0$.*

Definition 2.3 *[3] Let $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ be two fixed point iteration procedure sequences that converge to the same fixed point p and $d(u_n,p)\leq a_n$ and $d(v_n,p)\leq b_n$ for all $n\geq 0$. If the sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ converge to a and b, respectively, and $lim_{n\to\infty}\frac{d(a_n,a)}{d(b_n,b)}=0$, then we say that $\{u_n\}_{n=0}^{\infty}$ converge faster than $\{v_n\}_{n=0}^{\infty}$ to p.*

Definition 2.4 *[11] Let $\{t_n\}_{n=0}^{\infty}$ e an arbitrary sequence in C. Then, an iteration procedure $x_{n+1}=f(T,x_n)$ converging to fixed point p, is said to be T-stable or stable with respect to T, if for $\epsilon_n=d(t_n,f(T,t_n)),n=0,1,2,\ldots.,$ we have* $\lim_{n\to\infty}\epsilon_n=0$ if and only if $\lim_{n\to\infty}t_n=p$.

Lemma 2.5 *[23] Let $\{\psi_n\}_{n=0}^{\infty}$ and $\{\phi_n\}_{n=0}^{\infty}$ be nonnegative real sequences satisfying the following inequality:*

$$\psi_{n+1}\leq(1-\phi_n)\psi_n+\phi_n,$$

where $\phi_n\in(0,1)$, for all $n\in\mathbb{N},\sum_{n=0}^{\infty}\phi_n=\infty$ and $\frac{\phi_n}{\psi_n}\to 0$ as $n\to\infty$ then $\lim_{n\to\infty}\psi_n=0$."

3 Convergence analysis

$\{a_n\}$ and $\{b_n\}$ are real sequences in $[0,1]$, and T is any self map defined on a nonempty subset C of a CAT(0) space X in this section.

Gursoy and Karakaya [9] proposed the Picard-S iteration process, which is as follow:

$$\begin{aligned} &u_0\in C\\ &w_n=(1-b_n)u_n+b_nTu_n\\ &v_n=(1-a_n)Tu_n+a_nTw_n\\ &u_{n+1}=Tv_n. \end{aligned}$$

They show how to approach the fixed point of contraction maps using the Picard-S iteration process.

Thakur et al. [20] established a new iteration process to approximate invariant points, which they characterised as:

$$\begin{aligned} &u_0\in C\\ &w_n=(1-b_n)u_n+b_nTu_n\\ &v_n=T((1-a_n)u_n+a_nw_n)\\ &u_{n+1}=Tv_n. \end{aligned}$$

They demonstrated that their current iteration method is faster than some known iteration

processes for some types of maps using numerical examples.
Lamba and Panwar [13] recently developed approximaion results using a new iteration method, known as AP iteration process in the context of CAT(0) space, as follows:

$$x_0 \in T$$
$$z_n = T((1-b_n)x_n \oplus b_n T x_n)$$
$$y_n = T((1-a_n)Tx_n \oplus a_n T z_n)$$
$$x_{n+1} = Ty_n. \quad (3.1)$$

In comparison to previous iteration processes, we demonstrate that our AP iteration process is stable and has a high convergence rate.

Theorem 3.1 *Suppose T is a self contraction map defined on a nonempty closed convex subset of a complete CAT(0) space X. Also,* $\{x_n\}$ *is an iterative sequence formed by (3.1) with real sequences* $\{a_n\}_{n=0}^{\infty}$ *and* $\{b_n\}_{n=0}^{\infty}$ *in [0,1] that pleases* $\sum_{n=1}^{\infty} a_n b_n = \infty$. *Then the AP iterative scheme converges to the unique invariant point p of T.*

Proof. From [24], T has a unique a invariant point. We prove that $x_n \to p$ as $n \to \infty$. By (3.1), we get

$$\begin{aligned} d(z_n, p) &= d\left(T\left((1-b_n)x_n \oplus b_n T x_n\right), p\right) \\ &\leq \theta\left[d\left((1-b_n)x_n \oplus b_n T x_n, p\right)\right] \\ &\leq \theta\left[(1-b_n)d(x_n, p) + b_n d(Tx_n, p)\right] \\ &\leq \theta\left[(1-b_n)d(x_n, p) + \theta b_n d(x_n, p)\right] \\ &= \theta\left[1 - b_n(1-\theta)\right]d(x_n, p). \end{aligned}$$

Similarly,

$$\begin{aligned} d(y_n, p) &= d\left(T\left((1-a_n)T x_n \oplus a_n T z_n\right), p\right) \\ &\leq \theta\left[d\left((1-a_n)T x_n \oplus a_n T z_n, p\right)\right] \\ &\leq \theta\left[(1-a_n)d(T x_n, p) + a_n d(T z_n, p)\right] \\ &\leq \theta^2 \left[(1-a_n)d(x_n, p) + a_n d(z_n, p)\right] \\ &\leq \theta^2 \left[(1-a_n)d(x_n, p) + a_n \theta\left[1 - b_n(1-\theta)\right]d(x_n, p)\right] \\ &\leq \theta^2 \left[\left(1 - a_n b_n(1-\theta)\right)\right] d(x_n, p) \end{aligned}$$

Hence,

$$\begin{aligned} d(x_{n+1}, p) &= d(T y_n, p) \leq \theta d(y_n, p) \\ &\leq \theta^3 \left[1 - a_n b_n(1-\theta)\right] d(x_n, p) \end{aligned}$$

The inequalities that result from repeating the above steps as follow:

$$\begin{aligned} d(x_{n+1}, p) &\leq \theta^3 \left(1 - a_n b_n(1-\theta)\right)d(x_n, p) \\ d(x_n, p) &\leq \theta^3 \left(1 - a_{n-1}b_{n-1}(1-\theta)\right)d(x_{n-1}, p) \\ &. \\ &. \\ d(x_1, p) &\leq \theta^3 \left(1 - a_0 b_0(1-\theta)\right)d(x_0, p) \end{aligned}$$

It is simple to figure out that

$$d(x_n, p) \le d(x_0, p)\theta^{3(n+1)} \prod_{k=0}^{n}\left(1 - a_k b_k (1 - \theta)\right)$$

where $1 - a_n b_n(1-\theta) < 1$ because $\theta \in (0,1)$ and $a_n b_n \in [0,1]$ for all $n \in N$. We already know, $1 - x \le e^{-x}$ for all $x \in [0,1]$, hence we have

$$d(x_{n+1}, p) \le \frac{d(x_0, p)\theta^{3(n+1)}}{e^{(1-\theta)} \sum_{k=0}^{n} a_n b_n}$$

Taking the limit on both sides of above inequality yields $\lim_{n\to\infty} d(x_n, p) = 0$, i.e. $x_n \to p$ for $n \to \infty$, as necessary.

Theorem 3.2 *Assume C, X, T and* $\{x_n\}$ *are same as in Theorem 3.1. Then the AP iteration process is T-stable.*

Proof. Suppose $\{t_n\} \subset X$ is any arbitrary sequence in C. Assume the sequence formed by (3.1) is $x_{n+1} = f(T, x_n)$ converging to invariant point p and $\epsilon_n = d(t_{n+1}, f(T, z_n))$. we shall demonstrate $\lim_{n\to\infty} \epsilon_n = 0 \Leftrightarrow \lim_{n\to\infty} t_n = p$.

Consider $\lim_{n\to\infty} \epsilon_n = 0$. Using Theorem 4.1 we have

$$\begin{aligned} d(t_{n+1}, p) &\le d\left(t_{n+1}, f(T, t_n)\right) + d\left(f(T, t_n), p\right) \\ &= \epsilon_n + d(t_{n+1}, p) \\ &\le \epsilon_n + \theta^3\left(1 - a_n b_n (1 - \theta)\right) d(t_n, p) \end{aligned}$$

Describe $\psi = d(t_n, p), \phi = a_n b_n(1-\theta) \in (0,1)$ and $\varphi_n = \epsilon_n$, as $\theta \in (0,1), a_n, b_n \in [0,1]$, for all $n \in N$ and $\lim_{n\to\infty} \epsilon_n = 0$ which gives that conditions of Lemma 2.5 have been met. Therefore $\lim_{n\to\infty} d(t_n, p) = 0 \Longrightarrow \lim_{n\to\infty} t_n = p$.

For the converse part, suppose $\lim_{n\to\infty} t_n = p$, we get

$$\begin{aligned} \epsilon_n &= d\left(t_{n+1}, f(T, t_n)\right) \\ &\le d(t_{n+1} + p) + d\left(f(T, t_n), p\right) \\ &\le d(t_{n+1} + p) + \theta^3\left(1 - a_n b_n (1 - \theta)\right) d(t_n, p) \end{aligned}$$

Thus, $\lim_{n\to\infty} \epsilon_n = 0$

Therefore, (3.1) is T-stable.

Theorem 3.3 *Assume C, X and T are same as in Theorem 3.1. Suppose* $\{u_n\}$ *and* $\{x_n\}$ *are iterative sequences formed by Picard S iteration process and AP iteration process respectively fulfills (i)* $a \le a_n < 1$ *and* $b \le b_n < 1$, *for some* $a, b > 0$ *and* $\forall n \in N$ *(ii)* $\sum_{n=1}^{\infty} a_n b_n = \infty$. *Then the following are similar:*

(a) AP iteration process converges to the invariant point p of T.

(b) Picard S iteration process converges to the invariant point p of T.

Proof. we first demonstrate (i) ⇒ (ii). Assume Ap iteration method (3.1) converges to the invariant point p of T i.e. $\lim_{n\to\infty} d(x_n, p) = 0$. Now we have

$$\begin{aligned} d(z_n, w_n) &= d(T((1-b_n)x_n \oplus b_n T x_n), (1-b_n)u_n \oplus T u_n) \\ &\le \theta[d((1-b_n)x_n \oplus b_n T x_n, (1-b_n)u_n \oplus T u_n)] \\ &\le \theta[(1-b_n)d(x_n, (1-b_n)u_n \oplus T u_n), b_n d(T x_n, (1-b_n)u_n \oplus T u_n)] \\ &\le \theta[(1-b_n)^2 d(x_n, d(u_n) + (1-b_n)b_n d(T u_n), x_n) \\ &\quad + b_n(1-b_n)d(T x_n, u_n) + b_n^2 d(T x_n, T u_n)] \\ &\le \theta[(1-b_n)^2 d(x_n, d(u_n) + (1-b_n)b_n \theta d(u_n), x_n) \end{aligned}$$

$$+\theta b_n(1-b_n)d(x_n,u_n)+\quad b_n^2\theta d(x_n,u_n)]$$
$$\leq \theta(1-b_n(1-\theta))d(x_n,u_n)$$

Similarly, using Picard-S and AP iteartion method
we get $d(x_{n+1},u_{n+1})=d(Ty_n,u_{n+1})$

$$\leq d(Ty_n,y_n)+d(y_n,u_{n+1})$$
$$=d(Ty_n,y_n)+d(Tv_n,T((1-a_n)Tx_n\oplus a_nTz_n))$$
$$\leq d(Ty_n,y_n)+\theta[d(v_n,(1-a_n)Tx_n\oplus a_nTz_n)]$$
$$\leq d(Ty_n,y_n)+\theta[(1-a_n)d(v_n,Tx_n)+a_nd(Tz_n,v_n)]$$
$$\leq d(Ty_n,y_n)+\theta^2[(1-a_n)d(v_n,x_n)+a_nd(z_n,v_n)]$$
$$\leq d(Ty_n,y_n)+\theta^2[(1-a_n)d((1-a_n)Tu_n\oplus a_nTw_n,x_n)$$
$$+a_nd(z_n,(1-a_n)Tu_n\oplus a_nTw_n)]$$
$$\leq d(Ty_n,y_n)+\theta^2[(1-a_n)^2d(Tu_n,x_n)+(1-a_n)a_nd(Tw_n,x_n)$$
$$+a_n(1-a_n)d(z_n,Tu_n)+a_n^2d(Tw_n,z_n)]$$
$$\leq d(Ty_n,y_n)+\theta^2[(1-a_n)^2\theta d(u_n,x_n)+(1-a_n)a_n\theta d(w_n,x_n)$$
$$+a_n(1-a_n)\theta d(z_n,u_n)+a_n^2\theta d(w_n,z_n)]$$
$$\leq d(Ty_n,y_n)+\theta^2[(1-a_n)^2\theta d(u_n,x_n)+(1-a_n)a_n\theta d\big((1-b_n)u_n\oplus b_nTu_n,x_n\big)+$$
$$a_n(1-a_n)\theta d(u_n,(1-b_n)x_n\oplus b_nTx_n)+a_n^2\theta d(w_n,z_n)]$$
$$\leq d(Ty_n,y_n)+\theta^2[(1-a_n)^2\theta d(u_n,x_n)+(1-a_n)a_n(1-b_n)\theta d(u_n,x_n)$$
$$+(1-a_n)a_nb_n\theta d(u_n,x_n)+a_n(1-a_n)(1-b_n)\theta d(u_n,x_n)$$
$$+a_n(1-a_n)b_n\theta d(x_n,u_n)+a_n^2\theta^2(1-b_n(1-\theta))d(x_n,u_n)]$$

Since $\theta\in(0,1)$ and $\{a_n\},\{b_n\}\in[0,1]$, we have

$$d(x_{n+1},u_{n+1})\leq d(Ty_n,y_n)+(1-a_n(1-\theta))d(x_n,u_n)$$

Define $\psi_n=d(x_n,u_n),\phi_n=a_n(1-\theta)\in(0,1)$ and $\varphi=d(Ty_n,y_n)$.
Since $\lim_{n\to\infty}d(x_n,p)=0$ and $Tp=p$, so

$$\lim_{n\to\infty}d(Ty_n,y_n)=\lim_{n\to\infty}d(Ty_n,Tp)+d(Tp,y_n)$$
$$\leq(1+\theta)\lim_{n\to\infty}d(y_n,p)$$
$$=0,$$

which implies that all the conditions of Lemma 2.5 are fulfilled. Hence, we get

$$\lim_{n\to\infty}\psi_n=\lim_{n\to\infty}d(x_n,u_n)=0.$$

So, we get $d(u_n,p)\leq d(x_n,u_n)+d(x_n,p)\to 0$ as $n\to\infty$. So $\lim_{n\to\infty}d(u_n,p)=0$ i.e. the Picard-S iteration process converges to the inariant point of T.

Next we show $(ii)\Rightarrow(i)$. Suppose $\lim_{n\to\infty}d(u_n,p)=0$.

Now, we get, $\quad d(u_{n+1},x_{n+1})=d(Ty_n,Tv_n)$

$$\leq\theta d(y_n,v_n)$$
$$\leq\theta d\{T((1-a_n)Tx_n\oplus a_nTz_n),v_n\}$$
$$\leq\theta^2d\{(1-a_n)Tx_n\oplus a_nTz_n,v_n\}$$
$$\leq\theta^2\{(1-a_n)d(Tx_n,v_n)+a_nd(Tz_n,v_n)\}$$
$$\leq\theta^3\{(1-a_n)d(x_n,v_n)+a_nd(z_n,v_n)\}$$
$$\leq\theta^3(1-a_n(1-\theta))d(x_n,u_n)$$
$$\leq\theta^3(1-a_n(1-\theta))d(x_n,u_n)$$

Decribe $\psi_n=d(u_n,x_n),\phi_n=\theta^3(1-a_n(1-\theta))$ and $\varphi_n=0$.
As a result, all conditions of Lemma 2.5 are met and hence,

$$\lim_{n\to\infty}\psi_n=\lim_{n\to\infty}d(u_n,x_n)=0.$$

Using this we get $d(x_n,p)\leq d(u_n,x_n)+d(u_n,p)\to 0$ as $n\to\infty$. Hence $\lim_{n\to\infty}d(x_n,p)=0$ as required.

Theorem 3.4 *Assume C, X, T,* $\{u_n\}$ *and* $\{x_n\}$ *are same as in Theorem 3.3 pleasing (i)* $a \le a_n < 1$ *and* $b \le b_n < 1$, *for some* $a, b > 0$ *and for all* $n \in N$. *Then* $\{x_n\}$ *converge to p faster than* $\{u_n\}$ *does.*

Proof. The following inequality is due to [9] which is obtained from Picard S iteration process, also converging to unique invariant point p.

$$d(u_{n+1}, p) \le d(u_0, p)\theta^{2(n+1)} \prod_{k=0}^{n}\left(1 - a_k b_k (1 - \theta)\right)$$

$$\le d(u_0, p)\theta^{2(n+1)} \prod_{k=0}^{n}\left(1 - ab(1 - \theta)\right)$$

$$\le d(u_0, p)\theta^{2(n+1)} \left(1 - ab(1 - \theta)\right)^{n+1}$$

From Theorem 4.1 we have,

$$d(x_{n+1}, p) \le d(x_0, p)\theta^{3(n+1)} \prod_{k=0}^{n}\left(1 - a_k b_k (1 - \theta)\right)$$

$$\le d(x_0, p)\theta^{3(n+1)} \prod_{k=0}^{n}\left(1 - ab(1 - \theta)\right)$$

$$= d(x_0, p)\theta^{3(n+1)} \left(1 - ab(1 - \theta)\right)^{n+1}$$

Define $\alpha_n = d(x_0, p)\theta^{3(n+1)}(1 - ab(1 - \theta)$, and $b_n = d(u_0, p)\theta^{2(n+1)}(1 - ab(1 - \theta))^{n+1}$ then $\Psi = \frac{a_n}{b_n} = \frac{d(x_0,p)\theta^{3(n+1)}}{d(u_0,p)\theta^{2(n+1)}} = \theta^{n+1}$.

Since $\frac{\Psi_{n+1}}{\Psi_n} = \lim_{n\to\infty} \frac{\theta^{n+2}}{\theta^{n+1}} = \theta < 1$, hence by ratio test $\sum_{n=0}^{\infty} \Psi_n < \infty$.

So, $\lim_{n\to\infty} \frac{d(x_{n+1},p)}{d(u_{n+1},p)} = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \Psi_n = 0$

gives $\{x_n\}$ is faster than $\{u_n\}$.

The following numerical examples demonstrate the efficiency of the AP iteration method and support the analytical proof of Theorem. We can see right away the latest AP iteration method is the first to converge. The sequence of each iteration procedure is illustraed graphically, where sequence of each iteration procedure is denoted by x_n.

Example 3.5 *Consider* $T: [0,4] \to [0,4]$ *defined as* $T(x) = (x+2)^{\frac{1}{3}}$ *, be any map. T is a contraction map, as you can see. As a result, T has a unique invariant point 1.52137970680457.*

Table 1: **Iterative values of AP, Picard-S and Thakur iteration processes for the map** $\boldsymbol{T(x) = (x+2)^{\frac{1}{3}}}$**, where** $\boldsymbol{a_n = b_n = 0.25}$ **for all n**

n	AP	Picard-S	Thakur
1	1.99	1.99	1.99
2	1.528442036220800	1.53015960840678	1.530163435137870
3	1.521490093814170	1.52155190179242	1.521551978070990
4	1.521381433182300	1.52138308699161	1.521383088489430
5	1.521379733804180	1.52137977315880	1.521379773188200
6	1.521379707226830	1.52137970810712	1.521379708107700
7	1.521379706811170	1.52137970683014	1.521379706830150
8	1.521379706804670	1.52137970680507	1.521379706805070
9	1.521379706804570	1.52137970680458	1.521379706804580

10	1.521379706804570	1.52137970680457	1.521379706804570

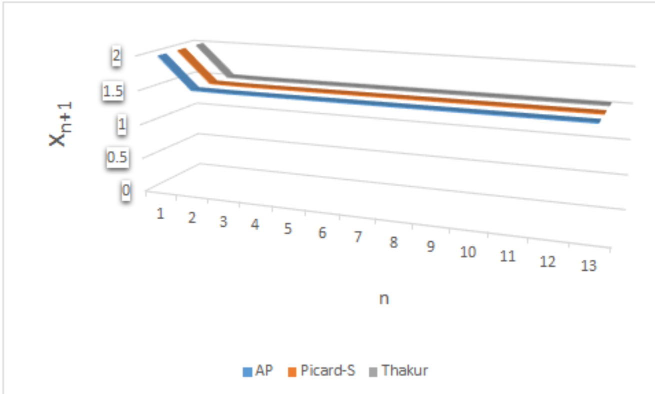

Figure 1: Convergence of AP, Picard-S and Thakur iterations to the invariant point 1.52137970680457 of mapping T.

Example 3.6 *Suppose* $T:[0,1] \to [0,1]$ *is any map defined as* $T(x) = \frac{x}{2}$. *T is a contraction map, as can be seen. So, T has only one invariant point 0.*

Table 2: **Iterative values of AP, Picard-S and Thakur iteration processes for** $\boldsymbol{a_n = b_n = 0.25}$**, for all n and the map** $\boldsymbol{Tx = \frac{x}{2}}$.

n	AP	Picard-S	Thakur
1	0.9	0.9	0.9
2	0.181054687500000	0.21796875000000	0.217968750000000
3	0.036423110961914	0.05278930664063	0.052789306640625
	0.007327305525541	0.01278491020203	0.012784910202026
4	0.001474047791271	0.00309634543955	0.003096345439553
5	0.000296536958010	0.00074989616114	0.000749896161142
6	.	.	.
7	.	.	.
16	3.21923E-11	5.20617E-10	5.20617E-10
17	6.47618E-12	1.26087E-10	1.26087E-10

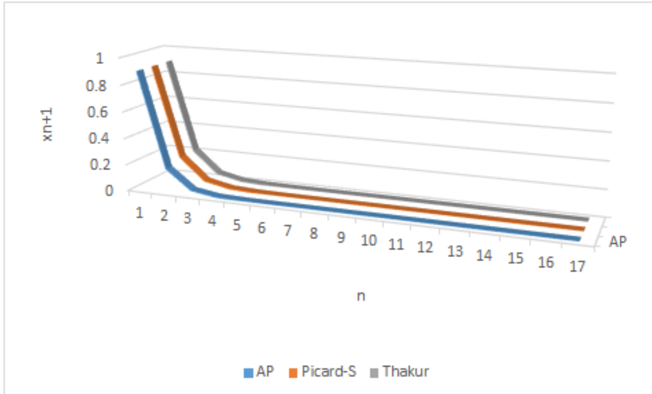

Figure 2: Convergence of AP, Picard-S and Thakur iterations to the invariant point 0 of mapping T.

4 Conclusion

Theorem 3.1 shows that, like other current iteration processes for contraction maps, our AP iteration process is also converges to a invariant point. We demonstrate in Theorem 3.4 that our AP iteration process is faster than the leading Picard-S iteration process,established by Gursoy and Karakaya [9]. The examples 3.5 and 3.6 are given to back up our argument. Data scientists, engineers, mathematicians, and physicists can now use the AP iteration process to solve various problems more effectively.

Conflict of Interest

The authors declare that there are no conflicts of interest.

References

[1] Abbas, M. and Nazir, T. (2014). A new faster iteration process applied to constrained minimization and feasibility problems. Mat Vesn, 66, 223–234.

[2] Agarwal, R. P., Regan, D. O. and Sahu, D. R. (2007). Iterative construction of fixed points of nearly asymptotically nonexpansive mappings. J Nonlinear Convex Anal, 8, 61–79.

[3] Berinde, V. (2007). Iterative approimation of fixed points. Springer, Berlin.

[4] Bhutia, J. D. and Tiwary, K. (2019). New iteration process for approximating fixed points in Banach spaces. Journal of Linear and Topological Algebra, 8(4), 237-250.

[5] Burago D., Burago Y., and Ivanov S. (2001). A Course in Metric Geometry. In:Graduate studies in Math., Amer. Math. Soc., Providence, Rhode Island.

[6] Bridson, M. R. and Haefliger, A. (1999). Metric spaces of non-positive curvature. Grundlehren der Mathematischen Wissenschaften, 319, Springer, Berlin.

[7] Chugh, R. Kumar, V. and Kumar, S. (2012). Strong convergence of a new three step iterative scheme in Banach spaces. American J Comp Math, 2, 345–357.

[8] Gromov, M. (1987). Hyperbolic Groups. Essays in group theory. Math. Sci. Res. Inst. Publ., 8, Springer, New York.

[9] Gursoy, F. and Karakaya, V. (2014). A Picard-S hybrid type iteration method for solving a differential equation with retarded argument. arXiv:1403.2546v2.

[10] Harder A. M. (1987). Fixed point theory and stability results for fixed points iteration procedures. Ph. D. Thesis, University of MissouriRolla.

[11] Harder A. M. and Hicks T. L. (1988). Stability results for fixed point iteration procedures. Math. Japonica, 33(5), 693–706.

[12] Ishikawa, S. (1974). Fixed points by a new iteration method. Proc. Am. Math. Soc., 44, 147-150.

[13] Lamba, P. and Panwar, A. On different results for new three step iteration process in CAT(0) space. Journal of Interdisciplinary Mathematics. doi: 10.1080/09720502.2020.1860285.

[14] Mann, W. R. (1953). Mean value methods in Iteration. Proc. Am. Math. Soc., 4, 506-510.

[15] Noor, M. A. (2000). New approximation schemes for general variational inequalities. J Math Anal Appl, 251, 217–229.

[16] Phuengrattana, W. and Suantai, S. (2011). On the rate of convergence of Mann, Ishikawa, Noor and SP-iterations for continuous functions on an arbitrary interval. J Comput Appl Math, 235, 3006–3014.

[17] Rhoades, B. E. (1991). Some fixed point iteration procedures. Int J Math Math Sci, 14, 1–16.

[18] Rhoades, B. E. (1977). Fixed point iterations using infinite matrices, III. Fixed points, algorithms and applications. Academic Press, Cambridge.

[19] Sahu, D. R. and Petrusel, A. (2011). Strong convergence of iterative methods by strictly pseudocontractive mappings in Banach spaces. Nonlinear Anal Theory Methods Appl, 74, 6012–6023.

[20] Thakur, B. S., Thakur, D. and Postolache, M. (2016). A new iterative scheme for numerical reckoning fixed points of Suzuki's generalized nonexpansive mappings. Appl. Math. Comp., 275, 147-155.

[21] Ullah, K. and Arshad, M. (2016). On different results for new three step iteration process in Banach spaces. Springer Plus, 5:1616.

[22] Ullah, K. and Arsahd, M. (2018). New three-step iteration process and fixed point approximation in Banach spaces. Journal of Linear and Topological Algebra, 7(2), 87-100.

[23] Weng, X. (1991). Fixed point iteration for local strictly pseudocontractive mapping. Proc Am Math Soc, 113, 727-731.

[24] Xue, Z. (2016). The convergence of fixed point for a kind of weak contraction. Nonlinear Func. Anal. Appl., 21(3), 497–500.