Turkish Online Journal of Qualitative Inquiry (TOJQI)
Volume 12, Issue 6, July, 2021: 9159 - 9166

Research Article

# Invariant point theorems with PPF dependence for some contractions

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## Abstract

In this paper, some results are concerning the existence and uniqueness of invariant point with PPF dependence of a non linear mapping for $\theta - \phi$ contraction and $\theta - \phi$ Suzuki contraction in the complete metric space and setting of complete metric spaces. The new feature of this work is that the domain and range space of operator are not identical in question. The results provided in this research are on PPF dependent invariant points which are broaden and expand invariant point results by Mohamed Jleli and Bassem Samet [6] and Dingwei Zheng, Zhangyong Cai and Pei Wang [5].

**MSC:** 47H10, 54H25.

***Keywords:*** Invariant point; Invariant point with PPF dependence; Existence and uniqueness; Complete metric space.

## 1. Introduction

There are well known problems with many branches of mathematical work in the form fx=x for self-mapping f to be transformed into a fixed point problem. Several mathematicians have been moving in various ways to modify the results by changing the space or extending a single value mapping to a multiple valuable mapping system. The map's contractive character is lessened in various generalisations; see ([7], [8], [9], [10]) and the topology is weakened in some other generalizations; see ([11], [12], [13], [14], [15], [16]). The concept of general metric spaces was introduced in 2000 by Branciari [12] where the inequality of triangles was substituted with $d(x,y) \leq d(x,u) + d(u,v) + d(v,y)$ for all pairwise different points $x, y, u, v \in X$. Several fixed point outcomes have been set up on these spaces. Mohamed Jleli and Bessem Samet [6] presented a novel form of contractive map and proved a new fixed point theory for these kind of maps just on setting of generalised metric spaces in 2014. In 2017, Dingwei Zheng, Zhangyong cai and pei Wang [5] introduced the notions of $\theta - \phi$ contraction and $\theta - \phi$ Suzuki contraction and establish some new fixed point theorems for these mappings in the setting of complete metric spaces.

On the other side Bernfeld et al. [1] introduced the concept of PPF (Past Present Future) dependent invariant points, which is one type of invariant point for nonself mapping. This type of fixed point results have distinct domain and range. In addition, they gave the concept of Banach type contraction for a non-self mapping and demonstrated the existence in the Razumikhin class of PPF dependent fixed point theorems for contraction mappings of the Banach type. Such findings are used to prove solutions to nonlinear functional differential and integral equations that are

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dependent on past history, present data and future considerations. Many researchers have demonstrated several fixed points with PPF dependence results (see [2], [3], [4]).

In this article, we extend and generalize the outcomes of Mohamed Jleli and Bessem Samet [6] and Dingwei Zheng, Zhangyong cai and pei Wang [5] and we will prove these results for existence and uniqueness of invariant point with PPF dependence in complete metric spaces and in the setting of complete metric spaces.

## 2.Preliminaries

Let E denote a metric space or a Banach space with the norm $\|.\|_E$. $E_0$ = C(I, E) represents the set of all continuous E-valued functions on I, where I represents a closed interval [a, b] in $\mathbb{R}$. Any function of $E_0$ equips with the supremum norm $\|.\|_{E_0}$ defined by

$$\| \phi \|_{E_0} = \sup_{d\in I} \| \phi(d) \|_E$$

“A point $\phi \in E_0$ is said to be **PPF dependent invariant point** or an invariant point with PPF dependence of a nonself mapping S: $E_0 \to E$

if $S(\phi) = \phi(c)$ for some $c \in I$."

**Definition 2.1**[12] “Let E be a non-empty set and $d: E \times E \to [0, \infty)$ be a mapping such that for all $x, y \in E$ and for all distinct points $u, v \in E$, each of them different from x and y, one has

(i) $d(x, y) = 0 \Leftrightarrow x = y$

(ii) $d(x, y) = d(y, x)$

(iii) $d(x, y) \le d(x, u) + d(u, v) + d(v, y)$

Then (E,d) is called a generalized metric space (for short G.M.S.)."

**Definition 2.2**[12] “Let (E, d) be a G.M.S., $x_n$ be a sequence in E and $x \in E$. We say that

(i) $x_n$ is convergent to x if and only if $d(x_n, x) \to 0$ as $n \to \infty$. We denote this by $x_n \to x$.

(ii) $x_n$ is Cauchy if and only if $d(x_n, x_m) \to 0$ as $m, n \to \infty$.

(iii) (E, d) is complete if and only if every Cauchy sequence in E converges to some element in E."

**Definition 2.3** [5] “Let (E, d) be a metric space and $S: E \to E$ be a mapping.

(1) S is said to be a $\theta - \phi$ contraction if there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that for any $x, y \in E$,

$$d(Sx, Sy) \neq 0 \Rightarrow \theta(d(Sx, Sy)) \le \phi[\theta(N(x, y))].$$

(2) S is said to be a $\theta - \phi$ Suzuki contraction if there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that for any $x, y \in E,\ Sx \neq Sy$

$$\frac{1}{2} d(x, Sx) < d(x, y) \Rightarrow \theta(d(Sx, Sy)) \le \phi[\theta(N(x, y))]$$

where N(x,y) = max {d(x,y), d(x,Sx), d(y,Sy)}

(3) S is said to be a $\theta - \phi$ Kannan-type contraction if there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that for any $x, y \in E, Sx \neq Sy$,

$$\theta(d(Sx, Sy)) \le \phi[\theta(\frac{d(x,Sx)+d(y,Sy)}{2})]$$

We can easily see that $\theta - \phi$ contractions and $\theta - \phi$ Kannan-type contractions are $\theta - \phi$ Suzuki contractions."

## 3 .Main Result

We take $\Theta$ as the family of functions $\theta: (0, \infty) \to (1, \infty)$ which satisfying the following conditions:

$(\Theta_1)$ $\theta$ is non-decreasing
$(\Theta_2)$ for every sequence $x_n \subset (0,\infty)$, $\lim_{n\to\infty}\theta(x_n) = 1$ iff $\lim_{n\to\infty}x_n = 0^+$
$(\Theta_3)$ $\exists$ $p \in (0,1)$ and $q \in (0,\infty]$ such that $\lim_{x\to 0^+}\frac{\theta(x)-1}{x^p} = q$.

**Theorem 3.1** Let $S: E_0 \to E$ be a nonself mapping, where E is a complete G.M.S. Let $\exists$ $\theta \in \Theta$ and $r \in (0,1)$, such as

$$\psi,\xi \in E_0,\ \ \| S\psi - S\xi \|_E \neq 0 \quad \Rightarrow \quad \theta(\| S\psi - S\xi \|_E) \leq [\theta(\| \psi - \xi \|_{E_0})]^r \tag{1}$$

Then S has a fixed point with PPF dependent, that is unique.
**Proof** Let $\psi_0$ be any element of $E_0$. Clearly $S(\psi_0) \in E$.
So $\exists$ $y_1 \in E$ such that $S(\psi_0) = y_1$
Choose $\psi_1 \in E_0$ as $y_1 = \psi_1(c)$
Now for $\psi_1 \in E_0$, we have $S\psi_1 \in E$.
This means $\exists$ $y_2 \in E$ such that $S\psi_1 = y_2$
So, we can choose $\psi_2 \in E_0$ as $y_2 = \psi_2(c)$.
Continuing like this, $S\psi_{n-1} = \psi_n(c)$ $\forall$ $n \in \mathbb{N}$.
Now

$$\| \psi_{n-1} - \psi_n \|_{E_0} = \| \psi_{n-1}(c) - \psi_n(c) \|_E \quad \forall n \in \mathbb{N}.$$

For some $p \in \mathbb{N}$ we will prove $S(\psi_p) = S(\psi_{p+1})$
Let if possible $\| S\psi_n - |S\psi_{n+1} \| > 0$ $n \in \mathbb{N}$
From (1), we get

$$\begin{aligned}\theta(\| S\psi_n - S\psi_{n+1} \|_E) &= \theta(\| S(S\psi_{n-1}) - S(S\psi_n) \|_E) \\ &\leq [\theta(\| S\psi_{n-1} - S\psi_n \|_E)]^r \\ &\leq [\theta(\| S\psi_{n-2} - S\psi_{n-1} \|_E)]^{r^2} \leq \ldots \leq \\ &\leq [\theta(\| \psi_0(c) - S\psi_0 \|_E)]^{r^n}\end{aligned}$$

Thus

$$1 \leq \theta(\| S\psi_n - S\psi_{n+1} \|) \leq [\theta\ \ \psi_0(c) - S\psi_0 \|_E]^{r^n}, \quad \forall n \in \mathbb{N} \tag{2}$$

Applying $n \to \infty$

$$\theta(\| S\psi_n - S\psi_{n+1} \|) \to 1$$

which implies by $(\Theta_2)$ that

$$\lim_{n\to\infty} \| S\psi_n - S\psi_{n+1} \| = 0 \tag{3}$$

Now by $\Theta_3$, $\exists$ $p \in (0,1)$ and $q \in (0,\infty]$ such that

$$\lim_{n\to\infty}\frac{\theta(\|S\psi_n - S\psi_{n+1}\|)-1}{(\|S\psi_n - S\psi_{n+1})^p} = q$$

Suppose q is finite and $B = \frac{q}{2} > 0$
So, there exist $n_0 \in \mathbb{N}$ such that

$$\left|\frac{\theta(\|S\psi_n - S\psi_{n+1}\|)-1}{(\|S\psi_n - S\psi_{n+1}\|)^p} - q\right| \leq B \quad \forall n \geq n_0$$

which implies

$$B \geq \frac{\theta((\|S\psi_n - S\psi_{n+1}\|)-1}{(\|S\psi_n - S\psi_{n+1}\|)^p} \geq q - B$$

Thus $n(\| S\psi_n - S\psi_{n+1} \|)^p \leq An[\theta(\| S\psi_n - S\psi_{n+1} \|) - 1]$, $\forall$ $n \geq n_0$ and for $B = \frac{1}{A}$
Now let $n = \infty$ and $B > 0$ be any arbitrary number. So, $\exists$ $n_0 \in \mathbb{N}$ such that

$$\frac{\theta(\|S\psi_n - S\psi_{n+1}\|)-1}{(\|S\psi_n - S\psi_{n+1}\|)^p} \geq B, \quad \forall\ n \geq n_0$$

That implies

$$n[\| S\psi_n - S\psi_{n+1} \|]^p \leq nA[\theta(\| S\psi_n - S\psi_{n+1} \|) - 1], \quad \forall \; n \geq n_0, \text{ where } A = \frac{1}{B}$$

Thus in both cases, $\exists \;\; n_0 \in \mathbb{N}$ and $A > 0$, such that

$$n[\| S\psi_n - S\psi_{n+1} \|]^p \leq nA[\theta(\| S\psi_n - S\psi_{n+1} \|) - 1], \quad \forall \; n \geq n_0$$

By (2)

$$n[\| S\psi_n - S\psi_{n+1} \|]^p \leq nA[\theta(\| \psi_0(c) - S\psi_0 \|)]^{r^n} - 1 \quad \forall \; n \geq n_0$$

For lim $n \to \infty$, we have

$$\lim_{n\to\infty} n(\| S\psi_n - S\psi_{n+1} \|)^r = 0$$

Thus $\exists \;\; n_1 \in \mathbb{N}$, such that

$$\| S\psi_n - S\psi_{n+1} \| \leq \frac{1}{n^{\frac{1}{p}}} \quad \forall \, n \geq n_1 \tag{4}$$

Now, assume that $\psi_{n-1} = \psi_n$ for some $n \in \mathbb{N}$.
This implies that $\psi_{n-1}(c) = \psi_n(c) = S\psi_{n-1}$.
So, S has a PPF dependent fixed point in $E_0$.
Let if possible $\psi_{n-1} \neq \psi_n \quad \forall \; n \in \mathbb{N}$.
So,

$$\begin{aligned}\theta\big(\| \psi_{n+1} - \psi_{n+3} \|_{E_0}\big) &= \theta(\| \psi_{n+1}(c) - \psi_{n+3}(c) \|_E) \\ &= \theta(\| S\psi_n - S\psi_{n+2} \|_E) \\ &\leq [\theta(\| S\psi_{n-1} - S\psi_{n+1} \|)]^r \\ &\leq [\theta(\| S\psi_{n-2} - S\psi_n \|)]^{r^2} \leq \ldots \\ &\leq [\theta(\| \psi_0(c) - S\psi_2 \|)]^{r^n}.\end{aligned}$$

As lim $n \to \infty$ and by $(\Theta_2)$

$$\lim_{n\to\infty} \| S\psi_n - S\psi_{n+2} \| = 0 \tag{5}$$

On the same way, by $(\Theta_2)$, $\exists \;\; n_2 \in \mathbb{N}$, such that

$\| S\psi_n - S\psi_{n+2} \| \leq \frac{1}{n^{\frac{1}{p}}} \quad \forall \quad n \geq n_2$ (6)

Now take $\max\{n_0, n_1\} = N$. We consider possible cases
(Case a) If $m > 2$ is an odd number, then we can write $m = 2z + 1$ for $z \in \mathbb{N}$.
Using (4), we have

$$\begin{aligned}\| S\psi_n - S\psi_{m+n} \| &\leq \| S\psi_n - S\psi_{n+1} \| + \| S\psi_{n+1} - S\psi_{n+2} \| + \ldots + \| S\psi_{2z+n} - S\psi_{2z+n+1} \| \\ &\leq \frac{1}{n^{\frac{1}{p}}} + \frac{1}{(n+1)^{\frac{1}{p}}} + \ldots + \frac{1}{(n+2z)^{\frac{1}{p}}} \leq \sum_{j=n}^{\infty} \frac{1}{j^{\frac{1}{p}}}\end{aligned}$$

(Case b) If $m > 2$ is an even number, then we can write $m = 2z$ for $z \in \mathbb{N}$.
Using (4) and (6), we get

$$\begin{aligned}\| S\psi_n - S\psi_{n+m} \| &\leq \| S\psi_n - S\psi_{n+2} \| + \| S\psi_{n+2} - S\psi_{n+3} \| + \ldots + \| S\psi_{n+2z-1} - S\psi_{n+2z} \| \\ &\leq \frac{1}{n^{\frac{1}{p}}} + \frac{1}{(n+2)^{\frac{1}{p}}} + \ldots + \frac{1}{(n+2z-1)^{\frac{1}{p}}} \leq \sum_{j=n}^{\infty} \frac{1}{j^{\frac{1}{p}}}\end{aligned}$$

Finally from all cases we have

$$\| S\psi_n - S\psi_{n+m} \| \leq \sum_{j=n}^{\infty} \frac{1}{j^{\frac{1}{p}}}, \quad \forall \quad n \in N, m \in \mathbb{N}$$

Now $\sum_{j=n}^{\infty} \frac{1}{j^{\frac{1}{p}}}$ is convergent . ($\because \frac{1}{p} > 1$).

That means $\{S\psi_n\}$ $i.e$ $\{\psi_{n+1}\}$ is a Cauchy sequence or we can say $\{\psi_n\}$ is Cauchy sequence.
Because of completeness , $\exists \;\; \psi^* \in E_0$ such that $\psi_n \to \psi^*$.
On the other side, we observe that S is continuous . If $S\psi \neq S\xi$, then by (1) we get

$$\log[\theta(\| S\psi - S\xi \|_E)] \leq r\log[\theta(\| \psi - \xi \|_{E_0})]$$

that implies $\| S\psi - S\xi \|_E \leq \| \psi - \xi \|_{E_0} = \| \psi(c) - \xi(c) \|_E \quad \forall \quad \psi, \xi \in E_0$.

By this observation we have $\| S\psi_{n+1} - S\psi^* \|_E \leq \| S\psi_n - \psi^*(c) \|_E$
As $n \to \infty$ , $S\psi_{n+1} \to S\psi^*$.
Now by [6], if $\lim_{n\to\infty} \| \psi_n - \psi \| = \lim_{n\to\infty} \| \psi_n - \xi \|$ then $\psi = \xi$.
Thus $S\psi^* = \psi(c)$ for some $c \in [a, b]$.
So $\psi^*$ is a PPF dependent fixed point of S.
Now we will prove that S has a unique PPF dependent fixed point.
Let if possible $\psi^*, \xi^* \in E_0$ are two PPF dependent fixed points of S, such that

$$\| \psi^*(c) - \xi^*(c) \| = \| S\psi^* - S\xi^* \| > 0.$$

From (1),

$$\begin{aligned}\theta(\| \psi^*(c) - \xi^*(c) \|) &= \theta(\| S\psi^* - S\xi^* \|)\\ &\leq [\theta(\| \psi^*(c) - \xi^*(c))]^r\\ &< \theta(\| \psi^*(c) - \xi^*(c) \|)\end{aligned}$$

which is a contradiction.
Thus S has unique PPF dependent fixed point.

**Corollary 3.2** Let $S: E_0 \to E$ be a mapping, where E is complete metric space. Let $\exists$, $\theta \in \Theta$ and $r \in (0,1)$ such as
for $\psi, \xi \in E_0$, $\| S\psi - S\xi \|_E \neq 0 \;\Rightarrow\; \theta(\| S\psi - S\xi \|_E \leq [\theta(\| \psi - \xi \|_{E_0})]^r$
Then S has unique PPF dependent fixed point.

**Proof** Since a metric space is a G.M.S. So, we instantly arrive to this conclusion.

Now we denote by $\Phi$ the set of functions $\phi: [1, \infty) \to [1, \infty)$ satisfying the following conditions:
$(\Phi_1)$ $\phi: [1, \infty) \to [1, \infty)$ is non-decreasing;
$(\Phi_2)$ for each $t > 1$, $\lim_{n\to\infty} \phi_n(t) = 1$;
$(\Phi_3)$ $\phi$ is continuous on $[1, \infty)$.

**Lemma 3.3** If $\phi \in \Phi$, then $\phi(1) = 1$ and $\phi(t) < t \;\; \forall \; t > 1$.

**Proof** Suppose, on the contrary, that $\exists$ $t_0 > 1$ such that $\phi(t_0) > t_0$. The monotonicity of $\phi(t)$ yields $\phi^n t_0 > t_0$ for each $n \in \mathbb{N}$, which is a contradiction to $\lim n \to \infty \phi^n t_0 = 1$. Thus $\phi(t) < t$ for each $t > 1$. Since $1 \leq \phi(1) \leq \phi(t) < t$ for each $t > 1$, passing to limit as $t \to 1$, we have $\phi(1) = 1$. Based on the function $\phi \in \Phi$, we give the following definition.

**Theorem 3.4** Let $(E_0, d)$ be a complete metric space and $S: E_0 \to E$ be a $\theta - \phi$ Suzuki contraction, i.e, $\exists$ $\theta \in \Theta$ and $\phi \in \Phi$ such as for any $\psi, \xi \in E_0$, $S\psi \neq S\xi$,

$$\frac{1}{2} \| \psi(c) - S\psi \|_E < \| \psi - \xi \|_{E_0} \;\Rightarrow\; \theta(\| S\psi - S\xi \|_E) \leq \phi[\theta(N(\psi, \xi))] \quad (1)$$

where

$$N(\psi, \xi) = \{\max \| \psi - \xi \|_E, \| \psi(c) - S\psi \|_E, \| \xi(c) - S\xi \|_E\}$$

Then S has unique fixed point with PPF dependance $\psi^* \in E_0$ such that the sequence $\psi_n$ converges to $\psi^*$ for all $\psi \in E_0$.

**Proof** Let $\psi_0$ be an arbitrary element of $E_0$. We define a sequence $\{\psi_n\}$ in $E_0$ such that $\psi_{n+1}(c) = S\psi_n$, $\forall$ $n \in \mathbb{N}$ where $c \in [a, b]$.
(case a) If $\psi_{n+1} = \psi_n$ for some $n \in \mathbb{N}$, then $\psi^* = \psi_n$ is a fixed point with PPF dependence for S.
(case b) If $\psi_{n+1} \neq \psi_n$ for all $n \in \mathbb{N}$, then $\| \psi_{n+1} - \psi_n \| \geq 0$ for all $n \in \mathbb{N}$.
So, $\frac{1}{2} \| \psi_n(c) - S\psi_n \| < \| \psi_n(c) - S\psi n \|_E$

By using (1) with $\psi_n = \psi$ and $S\psi_n = \xi$, we have

$$\theta(\| S\psi_n - S\psi_{n+1} \|_E) \leq \phi[\theta(N(\psi_n, \psi_{n+1}))] \tag{2}$$

where $N(\psi_n, \psi_{n+1}) = \max\{\| \psi_n - \psi_{n+1} \|_{E_0}, \| \psi_n(c) - S\psi_n \|_E, \| \psi_{n+1}(c) - S\psi_{n+1} \|_E\}$

$$= \max\{\| \psi_n - \psi_{n+1} \|_{E_0}, \| \psi_{n+1} - \psi_{n+2} \|_{E_0}\} \tag{3}$$

If $N(\psi_n, \psi_{n+1}) = |\psi_{n+1} - \psi_{n+2} \|_{E_0}$ for some $c \in [a, b]$

then by (2), we have

$$\theta(\psi_{n+1}, \psi_{n+2} \|_{E_0}) = \theta(\| S\psi_n - S\psi_{n+1} \|_E) \leq \phi[\theta(\| \psi_{n+1} - \psi_{n+2} \|_{E_0})]$$

which is a contradiction by lemma “ lemma"

So, by (3), $N(\psi_n, \psi_{n+1}) = \| \psi_n - \psi_{n+1} \|_{E_0}$.

From (2) we have,

$$\theta(\| S\psi_n - S\psi_{n+1} \|_E) \leq \phi[\theta(\| \psi_n - \psi_{n+1} \|_{E_0})]$$

Repeat this step again and again, we conclude

$$\theta(\| S\psi_{n-1} - S\psi_n \|_E) \leq \phi[\theta(\| \psi_{n-1} - \psi_n \|_{E_0})]$$
$$\leq \phi^2[\theta(\| \psi_{n-2} - \psi_{n-1} \|_{E_0})] \leq \ldots \leq$$
$$\leq \phi^n[\theta(\| \psi_0 - \psi_1 \|_{E_0})]$$

Now using definition of $\theta$ and property $\Phi_2$, we get

$$\lim_{n\to\infty} \phi^n[\theta(\| \psi_0 - \psi_1 \|_{E_0})] = 1$$

and $\lim_{n\to\infty} \| \psi_n - \psi_{n+1} \|_{E_0} = 0$ (4)

Now we will prove that $\{\psi_n\}$ is a Cauchy sequence in E, otherwise, $\exists \ \ n > 0$ and sequence $\{s_n\}$ and $\{t_n\}$ such that $\forall \ \ n \in \mathbb{N}$,

$$n < t_n < s_n \ , \ \ \| \psi_{s_n(c)} - \psi_{t_n(c)} \|_{E_0} \geq \eta$$

and $\| \psi_{s_n(c)-1} - \psi_{t_n(c)} \|_{E_0} \leq \eta$

So, $\eta \leq \| \psi_{s_n(c)} - \psi_{t_n(c)} \|_{E_0} \leq \| \psi_{s_n(c)} - \psi_{s_n(c)-1} \|_{E_0} + \| \psi_{s_n(c)-1} - \psi_{t_n(c)} \|_{E_0}$

$$\leq \eta + \| \psi_{s_n(c)} - \psi_{s_n(c)-1} \|_{E_0}$$

From this inequality and (2.4), we have

$$\lim_{n\to\infty} \| \psi_{s_n(c)} - \psi_{t_n(c)} \|_{E_0} = \eta$$

(5)

We know that

$$\| \psi_{s_n(c)+1} - \psi_{t_n(c)+1} \|_{E_0} - \| \psi_{s_n(c)} - \psi_{t_n(c)} \|_{E_0} \leq \| \psi_{s_n(c)} - \psi_{s_n(c)+1} \|_{E_0} + \| \psi_{t_n(c)} - \psi_{t_n(c)+1} \|_{E_0}$$

Now from (4), (5) and above inequality

$$\lim_{n\to\infty} \| \psi_{s_n(c)+1} - \psi_{t_n(c)+1} \|_{E_0} \tag{6}$$

By (1) with $\psi_{s_n(c)} = \psi$ and $\psi_{t_n(c)} = \xi$, we have

$$\theta\left(\| \psi_{s_n(c)+1} - \psi_{t_n(c)+1} \|_{E_0}\right) \leq \phi\left[\theta\left(\max\left\{\| \psi_{s_n(c)} - \psi_{t_n(c)} \|_{E_0}, \| \psi_{s_n(c)} - \psi_{s_n(c)+1} \|_{E_0}, \| \psi_{t_n(c)} - \psi_{t_n(c)+1} \|_{E_0}\right\}\right)\right]$$

Using (4), (5), (6),$(\Theta_3)$ and $(\Phi_3)$, with $\lim \ n \to \infty$, we have

$$\theta(\eta) \leq \phi[\theta(\eta)]$$

from lemma (3.3) $\theta(\eta) \leq \phi[\theta(\eta)] < \theta(\eta)$,

which is a contradiction. Hence $\{\psi_n\}$ is a Cauchy sequence in $E_0$. And $E_0$ is complete. So $\{\psi_n\}$ converges to $\psi^*$ for some $\psi^* \in E_0$.

Let $l \in \mathbb{N}$ be any fixed number, we claim

$$\frac{1}{2} \| \psi_l(c) - S\psi_l \|_E < \| \psi_l(c) - \psi^*(c) \|_E \text{ or } \frac{1}{2} \| S\psi_l - S^2\psi_l \|_E < \| S\psi_l - \psi^*(c) \|_E$$

If not, then

$$\frac{1}{2}\parallel \psi_l(c) - S\psi_l \parallel_E \geq \parallel \psi_l(c) - \psi^*(c) \parallel_E \text{ and } \frac{1}{2}\parallel S\psi_l - S^2\psi_l \parallel_E \geq \parallel S\psi_l - \psi^*(c) \parallel_E \quad (7)$$

So, $2 \parallel \psi_l(c) - \psi^*(c) \parallel_E \leq \parallel \psi_l(c) - S\psi_l \parallel_E \leq \parallel \psi_l(c) - \psi^*(c) \parallel_E + \parallel \psi^*(c) - S\psi_l \parallel_E$

that means

$$\parallel \psi_l - \psi^* \parallel_{E_0} \leq \parallel \psi^*(c) - S\psi_l \parallel_E \quad (8)$$

By (7) and (8) $\parallel \psi_l - \psi^* \parallel_{E_0} \leq \parallel \psi^*(c) - S\psi_l \parallel_E \leq \frac{1}{2}\parallel S\psi_l - S^2\psi_l \parallel_E$ (9)

Using (1) with $\psi = \psi_{l(n)}$, $\xi = S\psi_{l(n)}$, we get

$$\theta(\parallel S\psi_l - S^2\psi_l \parallel) \leq \phi[\theta(\max\{\parallel \psi_l(c) - S\psi_l \parallel, \parallel S\psi_l - S^2\psi_l \parallel\})]$$

By lemma (3.3) and definition of $\phi$ and $\theta$ ,

$$\parallel S\psi_l - S^2\psi_l \parallel < \parallel \psi_l(c) - S\psi_l \parallel \quad (10)$$

By (7), (9) and (10)

$$\parallel S\psi_l - S^2\psi_l \parallel < \parallel \psi_l(c) - S\psi_l \parallel \leq \parallel \psi_l - \psi^* \parallel + \parallel \psi^*(c) - S\psi_l \parallel$$

$$\leq \frac{1}{2}\parallel S\psi_l - S^2\psi_l \parallel + \frac{1}{2}\parallel S\psi_l - S^2\psi_l \parallel = \parallel S\psi_l - S^2\psi_l \parallel$$

which is a contradiction. So, we can say, for each $n \in \mathbb{N}$

$$\frac{1}{2}\parallel \psi_l(c) - S\psi_l \parallel < \parallel \psi_l - \psi^* \parallel \quad \text{or} \quad \frac{1}{2}\parallel S\psi_l - S^2\psi_l \parallel < \parallel S\psi_l - \psi^*(c) \parallel$$

Case(I) If $\exists$ a subsequence $\{l_k\}$ such that $\forall \ k \in \mathbb{N}$

$$\frac{1}{2}\parallel \psi_{l_k}(c) - S\psi_{l_k} \parallel_E < \parallel \psi_{l_k} - \psi^* \parallel_{E_0}$$

then

$$\theta(\parallel S\psi_{l_k} - S\psi^* \parallel_E) \leq \phi[\theta(\parallel \psi_{l_k} - \psi^* \parallel_{E_0})]$$

By definition of $\theta$ and $\phi$,

$$\lim_{k\to\infty} \parallel S\psi_{l_k} - S\psi^* \parallel_E = 0$$

So, $\parallel \psi^*(c) - S\psi^* \parallel_E = \lim_{k\to\infty} \parallel \psi_{l_{k+1}}(c) - S\psi^* \parallel_E = \lim_{k\to\infty} \parallel S\psi_{l_k} - S\psi^* \parallel_E = 0$

Case(II) If $\exists$ a subsequence $\{l_k\}$ such that $\forall \ k \in \mathbb{N}$

$$\frac{1}{2}\parallel S\psi_{l_k} - S^2\psi_{l_k} \parallel_E < \parallel S\psi_{l_k} - \psi^*(c) \parallel_E$$

then

$$\theta(\parallel S^2\psi_{l_k} - S\psi^* \parallel_E) \leq \phi[\theta(\parallel S\psi_{l_k} - \psi^*(c) \parallel_E)]$$

So by definition of $\theta$ and $\phi$,

$$\lim_{k\to\infty} \parallel S^2\psi_{l_k} - S\psi^* \parallel_E = 0$$

Therefore, $\parallel \psi^*(c) - S\psi^* \parallel_E = \lim_{k\to\infty} \parallel \psi_{l_{k+2}}(c) - S\psi^* \parallel_E = \lim_{k\to\infty} \parallel S^2\psi_{l_k} - S\psi^* \parallel_E = 0$

Thus $\psi^*$ is a PPF dependent invariant point of S.

Now we will prove that uniqueness of PPF dependent invariant point of S. Let if possible $\exists$ another PPF dependent invariant point of S such that

$$S\psi^* = \psi^*(c) \neq S\xi^* = \psi^*(c), \text{ then}$$

$$\parallel S\psi^* - S\xi^* \parallel_E = \parallel \psi^* - \xi^* \parallel_{E_0} > 0 \text{ and } \frac{1}{2}\parallel \psi^*(c) - S\psi^* \parallel_E < \parallel \psi^* - \xi^* \parallel_{E_0}$$

Now by (1),

$$\theta(\parallel \psi^* - \xi^* \parallel_{E_0} = \theta(\parallel S\psi^* - S\xi^* \parallel_E) \leq \phi[\theta(\parallel \psi^* - \xi^* \parallel_{E_0})] < \theta(\parallel \psi^* - \xi^* \parallel_{E_0})$$

which is not possible.

So, PPF dependent invariant point of S is unique.

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