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Research Article

Convergence Results for Fixed Point of Multivalued Nonexpansive Mappings in Sequence Spaces

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Abstract

The aim of this research article is to prove the existence of fixed points and to establish some convergence and approximation results for the fixed point of multivalued nonexpansive mappings for bounded sets in sequence space ℓ^∞. In the end, we provide two numerical examples to establish our results.

Keywords: Multivalued nonexpansive mappings, Fixed point, Sequence space ℓ^∞.

2010 MSC: 46B45, 47H09, 47H10.

1. Introduction

Some fixed point results for nonexpansive mapping were proved by Browder [1] in 1965. After that there is great progress in the field of fixed point theory of nonexpansive mappings. A large number of results are obtained by many researchers (see [4, 5, 8, 9]). But, in 2020, Xianbing Wu provided some examples to establish that nonexpansive mappings have fixed points for some nonclosed bounded sets in Banach spaces whereas there are some nonexpansive mappings which have no fixed points for closed bounded sets.

Example 1.1. [9] “Let the set $C=\{-1,1\}$ and take $Tx=x$. Then $T:C\to C$ is a nonexpansive mapping and C is a closed bounded set. But T has no fixed in C.”

Example 1.2. [9] “Let the set $C=[0,1]$ and

$$Tx=\begin{cases}\frac{1}{4}, \text{if } x=1\\ \frac{7}{8}, \text{if } x\neq 1.\end{cases}$$

But $T:C\to C$ is not nonexpansive mapping in closed bounded set C. However, it is a nonexapansive mapping in nonclosed bounded set [0,1) and there exists a fixed point $\frac{7}{8}\in C=[0,1)$.”

He also proved some convergence results for the fixed point of nonexpansive mappings for bounded sets in sequence space ℓ^∞.

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2. Preliminaries

In the following section, some needed results and definitions are provided to make this paper self contained.

"Let X be a Banach space. Then a nonempty subset E of X is said to be proximinal if for each $x \in X$, there exists some $y \in E$ such that

$$\|x-y\| = \mathrm{dist}(x,E) = \inf\{\|x-y\| : y \in E\}.$$

Let P(E) denotes the family of all nonempty bounded proximinal subsets of E. Let $H(\cdot,\cdot)$ be the Hausdroff distance on P(E) defined as:

$$H(A,B) = \max\{\sup_{a\in A} \mathrm{dist}(a,B), \sup_{b\in B} \mathrm{dist}(b,A)\},\ A,B \in P(E)$$

where $\mathrm{dist}(a,B) = \inf\{\|a-b\| : b \in B\}$ denotes the distance from the point 'a' to the set B.

A multivalued mapping $T : E \to P(E)$ is said to be nonexpansive if

$$H(Tx,Ty) \le \|x-y\|, x, y \in E.$$

Lemma 2.1. [6] Let $T : E \to P(E)$ be multivalued mapping and $P_T(x) = \{y \in E : \|x-y\| = d(x,Tx)\}$. Then the following conditions are equivalent:

(1) $x \in F(T)$, i.e., $x \in Tx$;

(2) $P_T(x) = \{x\}$, i.e., $x = y$ for each $y \in P_T(x)$;

(3) $x \in F(P_T)$, i.e., $x \in P_T(x)$. Further, $F(T) = F(P_T)$.

Definition 2.2 [9] Let ψ be a bounded linear functional on ℓ^∞. Then ψ is called a Banach limit if it satisfies $\|\psi\| = \psi(1) = 1$ and $\psi(\beta_{n+1}) = \psi(\beta_n), \beta_n \in \ell^\infty$. Moreover, suppose ψ is a Banach limit, then the following conditions hold:

(i) If for all $n \in \mathbb{N}, \alpha_n \le \beta_n$ means $\psi(\alpha_n) \le \psi(\beta_n)$, where $\alpha_n, \beta_n \in \ell^\infty$

(ii) For each $n \in \mathbb{N}, \beta_n \in \ell^\infty$, we have $\psi(\beta_n) = \psi(\beta_{n+p})$

(iii) $\liminf_{n\to\infty} \beta_n \le \psi(\beta_n) \le \limsup_{n\to\infty} \beta_n$, where $\beta_n \in \ell^\infty$.

Lemma 2.3. [8] Suppose that $\{\alpha_n\}, \{\beta_n\}$ are two sequences of nonnegative numbers and $\sum \beta_n < \infty$, if there exists some number $N_0 \in \mathbb{N}$, for all $n \ge N_0$ such that $\alpha_{n+1} \le \alpha_n + \beta_n$,

then $\lim_{n\to\infty} \alpha_n$ exists."

3. Convergence Results in ℓ^∞

Theorem 3.1. Suppose $E \neq \phi$ is a bounded subset of sequence space ℓ^{∞} and $T : E \to P(E)$ is a multivalued mapping such that P_T is a nonexpansive mapping. If the following conditions hold:

(a) there is a sequence $\{\alpha_n\}$ in [0,1) such that $\lim_{n\to\infty} \alpha_n = 1, \sum \frac{|\alpha_{n+q} - \alpha_n|}{1-\alpha_n} < \infty,$ for all $n, q \in \mathbb{N}$ and $u \in E$ then $\alpha_n u \in E$

(b) if $\{u_n\}$ is a sequence in E and $\lim_{k\to\infty} u_{n_k} = u^*$ then $u^* \in E$

Then,

(i) T has atleast one fixed point in E

(ii) take $u_{n+1} = \alpha_n w_n$, where $w_n \in P_T(u_n)$ then the sequence $\{u_n\}$ converges to a fixed point of T.

Proof. Suppose $u_0 \in E$ and taking $u_{n+1} = \alpha_n w_n$, where $w_n \in P_T(u_n)$ and $\{\alpha_n\}$ is a sequence in [0,1) which satisfies condition (a) of theorem 3.1 which implies $u_{n+1} \in E.$ Now we prove that $\{u_n\}$ has a convergent subsequence $\{u_{n_k}\}$.

As P_T is a nonexpansive mapping, therefore for all $n, q \in \mathbb{N}$, we get

$$\begin{aligned}
\|u_{n+q} - u_n\| &= \|\alpha_{n+q-1} w_{n+q-1} - \alpha_{n-1} w_{n-1}\| \\
&= \|\alpha_{n+q-1} w_{n+q-1} - \alpha_{n-1} w_{n-1} + \alpha_{n-1} w_{n+q-1} - \alpha_{n-1} w_{n+q-1}\| \\
&\leq |\alpha_{n+q-1} - \alpha_{n-1}| \cdot \|w_{n+q-1}\| + \alpha_{n-1} \|w_{n+q-1} - w_{n-1}\| \\
&\leq |\alpha_{n+q-1} - \alpha_{n-1}| \cdot \|w_{n+q-1}\| + \alpha_{n-1} H(P_T(u_{n+q-1}), P_T(u_{n-1})))
\end{aligned}$$

$$\|u_{n+q} - u_n\| \leq |\alpha_{n+q-1} - \alpha_{n-1}| \cdot \|w_{n+q-1}\| + \alpha_{n-1} \|u_{n+q-1} - u_{n-1}\| \tag{3.1}$$

With help of Banach limit, we obtain

$$\begin{aligned}
\psi\left(\|u_{n+q} - u_n\|\right) &\leq \psi\left(|\alpha_{n+q-1} - \alpha_{n-1}| \cdot \|w_{n+q-1}\| + \alpha_{n-1} \|u_{n+q-1} - u_{n-1}\|\right) \\
&\leq |\alpha_{n+q-1} - \alpha_{n-1}| \cdot \psi\left(\|w_{n+q-1}\|\right) + \alpha_{n-1} \psi\left(\|u_{n+q-1} - u_{n-1}\|\right)
\end{aligned}$$

$$\psi\left(\|u_{n+q} - u_n\|\right) \leq |\alpha_{n+q-1} - \alpha_{n-1}| \cdot \psi\left(\|w_{n+q-1}\|\right) + \alpha_{n-1} \psi\left(\|u_{n+q-1} - u_{n-1}\|\right) \tag{3.2}$$

Due to boundedness of set E, there must exists a constant $L > 0$ such that $\|u\| \leq L$ for each $u \in E.$ From inequality (3.2), we obtain the following inequality

$$\psi\left(\|u_{n+q} - u_n\|\right) \leq \frac{|\alpha_{n+q-1} - \alpha_{n-1}|}{1-\alpha_{n-1}} \cdot \psi(L). \tag{3.3}$$

From condition (a) of theorem 3.1, we get

$$\lim_{n\to\infty} \frac{|\alpha_{n+q-1}-\alpha_{n-1}|}{1-\alpha_{n-1}} = 0.$$

Letting $n \to \infty$ in inequality (3.2), we get

$$\lim_{n\to\infty} \psi\left(\left\|u_{n+q}-u_n\right\|\right) = 0$$

or we can write

$$\lim_{m,n\to\infty} \psi\left(\left\|u_m-u_n\right\|\right) = 0 \text{ for } m,n \in \mathbb{N}.$$

Therefore, we have

$$0 \le \lim_{n\to\infty} \inf \left\|u_m-u_n\right\| \le \psi\left(\left\|u_m-u_n\right\|\right) = 0, \tag{3.4}$$

and hence, there must be a monotonically increasing sequence $\{n_k\}$ such that

$$\lim_{k\to\infty} \left\|u_{m_k}-u_{n_k}\right\| = 0,\ m,n \in \mathbb{N}. \tag{3.5}$$

So, $\{u_{n_k}\}$ is a Cauchy sequence, therefore, there exists some u^* such that

$$\lim_{k\to\infty} u_{n_k} = u^*. \tag{3.6}$$

Condition (b) of theorem 3.1 implies that $u^* \in E.$ Now, we show that $\{u_n\}$ converges to u^* and $u^* \in F(T).$ Using inequality (3.1), we get

$$\left\|u_{n+q}-u_n\right\| \le \left\|u_{n+q-1}-u_{n-1}\right\| + |\alpha_{n+q-1}-\alpha_{n-1}| \cdot \left\|w_{n+q-1}\right\| \tag{3.7}$$

With the help of condition (a) of theorem 3.1, we can write $\sum|\alpha_{n+q-1}-\alpha_{n-1}| < \infty$ and applying lemma 2.3 in inequality (3.7), $\lim_{k\to\infty}\left\|u_{n+q}-u_n\right\|$ exists. From equation (3.5), $\lim_{k\to\infty}\left\|u_{m_k}-u_{n_k}\right\| = 0,\ m,n \in \mathbb{N}.$ Therefore,

$$\lim_{m,n\to\infty}\left\|u_m-u_n\right\| = 0 \text{ for } m,n \in \mathbb{N}.$$

Hence, $\{u_n\}$ is a Cauchy sequence and using (3.6), we can write

$$\lim_{n\to\infty} u_n = u^*. \tag{3.8}$$

Now, we establish that u^* is a fixed point of T. Let $w^* \in P_T(u^*)$

$$\begin{aligned}\|u^* - w^*\| &= \|u^* - u_n + u_n - w^*\| \\ &\le \|u^* - u_n\| + \|u_n - w^*\| \\ &\le \|u^* - u_n\| + \|\alpha_{n-1} w_{n-1} - w^*\| \\ &\le \|u^* - u_n\| + \|\alpha_{n-1} w_{n-1} - \alpha_{n-1} w^* - (1-\alpha_{n-1}) w^*\| \\ &\le \|u^* - u_n\| + \alpha_{n-1}\|w_{n-1} - w^*\| + (1-\alpha_{n-1})\|w^*\| \\ &\le \|u^* - u_n\| + \alpha_{n-1} H(P_T(u_{n-1}), P_T(u^*)) + (1-\alpha_{n-1})\|w^*\| \\ &\le \|u^* - u_n\| + \alpha_{n-1}\|u_{n-1} - u^*\| + (1-\alpha_{n-1})\|w^*\|\end{aligned}$$

Using equation (3.8) and condition (a) of theorem 3.1, taking $n \to \infty$ in above inequality, we get $\|u^* - w^*\| = 0,$ that is, $u^* = w^*.$ Since $w^* \in P_T(u^*)$ was arbitrary, therefore, $P_T(u^*) = \{u^*\}$ which implies that u^* is a fixed point of T.

Corollary 3.2. Suppose $E \ne \phi$ is a bounded, closed subset of sequence space ℓ^∞ and $T : E \to P(E)$ is a multivalued mapping such that P_T is a nonexpansive mapping. If there is a sequence $\{\alpha_n\}$ in [0,1) satisfying $\lim_{n\to\infty} \alpha_n = 1, \sum \frac{|\alpha_{n+q} - \alpha_n|}{1-\alpha_n} < \infty,$ for all $n, q \in \mathbb{N}$ such that $\alpha_n u \in E$ for $u \in E.$ Then

(a) T has atleast one fixed point in E
(b) take $u_{n+1} = \alpha_n w_n,$ where $w_n \in P_T(u_n)$ then the sequence $\{u_n\}$ converges to a fixed point of T.

Theorem 3.3. Suppose $E \ne \phi$ is a bounded subset of sequence space ℓ^∞ and $T : E \to P(E)$ is a multivalued mapping such that P_T is a nonexpansive mapping. If the following two conditions hold:

(a) there is some $u_0 \in E$ and a sequence $\{\alpha_n\}$ in [0,1) satisfying $\lim_{n\to\infty} \alpha_n = 1, \sum \frac{|\alpha_{n+q} - \alpha_n|}{1-\alpha_n} < \infty,$ for all $n, q \in \mathbb{N}$ such that $(1-\alpha_n)u_0 + \alpha_n u \in E$ for $u \in E$
(b) if $\{u_n\}$ is a sequence in E and $\lim_{k\to\infty} u_{n_k} = u^*$ then $u^* \in E$

Then,

(i) T has atleast one fixed point in E
(ii) take $u_{n+1} = (1-\alpha_n)u_0 + \alpha_n w_n,$ where $w_n \in P_T(u_n)$ then the iterative sequence $\{u_n\}$ converges to a fixed point of T.

Proof. Suppose $u_0 \in E$ and taking $u_{n+1} = (1-\alpha_n)u_0 + \alpha_n w_n$, where $w_n \in P_T(u_n)$ and $\{\alpha_n\}$ is a sequence in [0,1) which satisfies condition (a) of theorem 3.3 which implies $u_{n+1} \in E$. Now we prove that $\{u_n\}$ has a convergent subsequence $\{u_{n_k}\}$.

As P_T is a nonexpansive mapping, therefore for all $n, q \in \mathbb{N}$, we get

$$\begin{aligned}
\left\|u_{n+q} - u_n\right\| &= \left\|(1-\alpha_{n+q-1})u_0 + \alpha_{n+q-1}w_{n+q-1} - (1-\alpha_{n-1})u_0 - \alpha_{n-1}w_{n-1}\right\| \\
&= \left\|(\alpha_{n-1} - \alpha_{n+q-1})u_0 + \alpha_{n+q-1}w_{n+q-1} - \alpha_{n-1}w_{n-1} + \alpha_{n-1}w_{n+q-1} - \alpha_{n-1}w_{n+q-1}\right\| \\
&\leq |\alpha_{n-1} - \alpha_{n+q-1}| \cdot \left\|u_0\right\| + |\alpha_{n+q-1} - \alpha_{n-1}| \cdot \left\|w_{n+q-1}\right\| + \alpha_{n-1}\left\|w_{n+q-1} - w_{n-1}\right\| \\
&\leq |\alpha_{n+q-1} - \alpha_{n-1}| \left(\left\|u_0\right\| + \left\|w_{n+q-1}\right\|\right) + \alpha_{n-1}H(P_T(u_{n+q-1}), P_T(u_{n-1})))
\end{aligned}$$

$$\left\|u_{n+q} - u_n\right\| \leq |\alpha_{n+q-1} - \alpha_{n-1}| \left(\left\|u_0\right\| + \left\|w_{n+q-1}\right\|\right) + \alpha_{n-1}\left\|u_{n+q-1} - u_{n-1}\right\| \tag{3.9}$$

With help of Banach limit, we obtain

$$\begin{aligned}
\psi\left(\left\|u_{n+q} - u_n\right\|\right) &\leq \psi\left(|\alpha_{n+q-1} - \alpha_{n-1}| \left(\left\|u_0\right\| + \left\|w_{n+q-1}\right\|\right) + \alpha_{n-1}\left\|u_{n+q-1} - u_{n-1}\right\|\right) \\
&\leq |\alpha_{n+q-1} - \alpha_{n-1}| \, \psi\left(\left\|u_0\right\| + \left\|w_{n+q-1}\right\|\right) + \alpha_{n-1}\psi\left(\left\|u_{n+q-1} - u_{n-1}\right\|\right)
\end{aligned}$$

$$\psi\left(\left\|u_{n+q} - u_n\right\|\right) \leq |\alpha_{n+q-1} - \alpha_{n-1}| \cdot \psi\left(\left\|u_0\right\| + \left\|w_{n+q-1}\right\|\right) + \alpha_{n-1}\psi\left(\left\|u_{n+q-1} - u_{n-1}\right\|\right) \tag{3.10}$$

Due to boundedness of set E, there must exists a constant $L > 0$ such that $\left\|u\right\| \leq L$ for each $u \in E$. From inequality (3.10), we get the following inequality

$$\psi\left(\left\|u_{n+q} - u_n\right\|\right) \leq \frac{|\alpha_{n+q-1} - \alpha_{n-1}|}{1-\alpha_{n-1}} \cdot \psi(2L). \tag{3.11}$$

From condition (a) of theorem 3.3, we get

$$\lim_{n\to\infty} \frac{|\alpha_{n+q-1} - \alpha_{n-1}|}{1-\alpha_{n-1}} = 0.$$

Letting $n \to \infty$ in inequality (3.11), we get

$$\lim_{n\to\infty} \psi\left(\left\|u_{n+q} - u_n\right\|\right) = 0$$

or we can write

$$\lim_{m,n\to\infty} \psi\left(\left\|u_m - u_n\right\|\right) = 0 \text{ for } m, n \in \mathbb{N}.$$

Therefore, we have

$$0 \le \lim_{n\to\infty} \inf \|u_m - u_n\| \le \psi\left(\|u_m - u_n\|\right) = 0, \tag{3.12}$$

means there must be a monotonically increasing sequence $\{n_k\}$ such that

$$\lim_{k\to\infty} \left\|u_{m_k} - u_{n_k}\right\| = 0,\ m, n \in \mathbb{N}. \tag{3.13}$$

So, $\{u_{n_k}\}$ is a Cauchy sequence, therefore, there exists some u^* such that

$$\lim_{k\to\infty} u_{n_k} = u^*. \tag{3.14}$$

Condition (b) of theorem 3.3 implies that $u^* \in E.$ Now, we show that $\{u_n\}$ converges to u^* and $u^* \in F(T).$ Using inequality (3.9), we get

$$\|u_{n+q} - u_n\| \le \|u_{n+q-1} - u_{n-1}\| + |\alpha_{n+q-1} - \alpha_{n-1}|\left(\|u_0\| + \|w_{n+q-1}\|\right). \tag{3.15}$$

With the help of condition (a) of theorem 3.3, we can write $\sum |\alpha_{n+q-1} - \alpha_{n-1}| < \infty$ and applying lemma 2.3 in inequality (3.15), $\lim_{k\to\infty} \|u_{n+q} - u_n\|$ exists. From equation (3.13), $\lim_{k\to\infty} \left\|u_{m_k} - u_{n_k}\right\| = 0,\ m, n \in \mathbb{N}.$ Therefore,

$$\lim_{m,n\to\infty} \|u_m - u_n\| = 0 \text{ for } m, n \in \mathbb{N}.$$

Hence, $\{u_n\}$ is a Cauchy sequence and using (3.14), we can write

$$\lim_{n\to\infty} u_n = u^*. \tag{3.16}$$

Now, we show that u^* is a fixed point of T. Let $w^* \in P_T(u^*)$

$$\begin{aligned}
\|u^* - w^*\| &= \|u^* - u_n + u_n - w^*\| \\
&\le \|u^* - u_n\| + \|u_n - w^*\| \\
&\le \|u^* - u_n\| + \|(1-\alpha_{n-1})u_0 + \alpha_{n-1}w_{n-1} - w^*\| \\
&\le \|u^* - u_n\| + \|(1-\alpha_{n-1})u_0 + \alpha_{n-1}w_{n-1} - \alpha_{n-1}w^* - (1-\alpha_{n-1})w^*\| \\
&\le \|u^* - u_n\| + \alpha_{n-1}\|w_{n-1} - w^*\| + (1-\alpha_{n-1})\|u_0 - w^*\| \\
&\le \|u^* - u_n\| + \alpha_{n-1}H(P_T(u_{n-1}), P_T(u^*)) + (1-\alpha_{n-1})\|u_0 - w^*\| \\
&\le \|u^* - u_n\| + \alpha_{n-1}\|u_{n-1} - u^*\| + (1-\alpha_{n-1})\|u_0 - w^*\|
\end{aligned}$$

Using equation (3.16) and condition (a) of theorem 3.3, taking $n \to \infty$ in above inequality, we get $\|u^* - w^*\| = 0,$ that is, $u^* = w^*.$ Since $w^* \in P_T(u^*)$ was arbitrary, therefore, $P_T(u^*) = \{u^*\}$ which that implies that u^* is a fixed point of T.

Corollary 3.4. Suppose $E \neq \phi$ is a bounded, closed subset of sequence space ℓ^{∞} and $T : E \to P(E)$ is a multivalued mapping such that P_T is a nonexpansive mapping. If there is some $u_0 \in E$ and a sequence $\{\alpha_n\}$ in [0, 1) satisfying $\lim_{n\to\infty} \alpha_n = 1, \sum \frac{|\alpha_{n+q} - \alpha_n|}{1-\alpha_n} < \infty,$ for all $n, q \in \mathbb{N}$ such that $(1-\alpha_n)u_0 + \alpha_n u \in E$ for $u \in E.$ Then

(a) T has atleast one fixed point in E
(b) take $u_{n+1} = (1-\alpha_n)u_0 + \alpha_n w_n$ where $w_n \in P_T(u_n)$ then the sequence $\{u_n\}$ converges to a fixed point of T.

Corollary 3.4. If $E \neq \phi$ is a bounded, closed and convex subset of sequence space ℓ^{∞} and $T : E \to P(E)$ is a multivalued mapping such that P_T is a nonexpansive mapping then T has a fixed point.

4. Numerical Examples

In this section, we provide illustrative examples to support our results.

Example 4.1. Consider a set $E = [0,1].$ Let $T : E \to P(E)$ be a multivalued mapping defined as:

$$T(u) = \left[0, \frac{u}{2}\right].$$

Clearly. E is bounded closed set. Define a sequence $\alpha_n = 1 - \frac{1}{(n+1)^2}$ we have $\{\alpha_n\}$ is a sequence in [0,1) which satisfies the following conditions for all $n, q \in \mathbb{N}$ **:**

(i) $\lim_{n\to\infty} \alpha_n = 1$

(ii) $\sum \frac{|\alpha_{n+q} - \alpha_n|}{1-\alpha_n} < \infty$

(iii) $\alpha_n u \in E$ for all $u \in E.$

As $P_T(u) = \{u\}$ for $u \in F(T)$ and if $u \notin F(T) = \{0\}$ then

$$P_T(u)=\left\{v\in T(u):\|u-v\|=d(u,Tu)=d\left(u,\left[0,\frac{u}{2}\right]\right)\right\}$$

$$=\left\{v\in T(u):\|u-v\|=\left\|u-\frac{u}{2}\right\|\right\}$$

$$=\left\{v\in T(u):u-v=\frac{u}{2}\right\}\text{ because } u>v\ \forall\, v\in T(u)\text{ where } u\in(0,1]$$

$$P_T(u)=\left\{v=\frac{u}{2}\right\}$$

Now, we show that P_T is nonexpansive mapping for all $u\in E.$ Let $u\in E$

$$H(P_T(u),P_T(p^*))=H\left(\frac{u}{2},p^*\right)=\left|\frac{u}{2}-p^*\right|\le\left|u-p^*\right|.$$

Above inequality establish that P_T is nonexpansive mapping for all $u\in E.$

As all the conditions of corollary 3.2 are satisfied, therefore there is atleast one fixed point of T. Now with the help of an iterative algorithm, we compute the fixed point of T. Taking $u_1=1$ and $u_{n+1}=\alpha_n w_n.$

$$u_2=\frac{3}{2}\cdot\frac{1}{2},u_3=\frac{2}{3}\cdot\frac{1}{2^2},u_4=\frac{5}{8}\cdot\frac{1}{2^3},u_5=\frac{3}{5}\cdot\frac{1}{2^4},\cdots,u_n=\frac{n+1}{2n}\cdot\frac{1}{2^{n-1}}$$

Tanking $n\to\infty,$ we get $\lim_{n\to\infty}u_n=0,$ which is the fixed point of T.

Example 4.2 Consider a set $E=[0,4].$ Let $T:E\to P(E)$ be a multivalued mapping defined as:

$$T(u)=\left[0,\frac{u+2}{2}\right].$$

Clearly, E is bounded closed set. Define a sequence $\alpha_n=1-\dfrac{1}{n+1}$ we have $\{\alpha_n\}$ is a sequence in [0, 1) which satisfies the following conditions for all $n,q\in\mathbb{N}$:

(i) $\lim_{n\to\infty}\alpha_n=1$

(ii) $\sum\dfrac{|\alpha_{n+q}-\alpha_n|}{1-\alpha_n}<\infty$

(iii) $(1-\alpha_n)u_1+\alpha_n u\in E$ for all $u\in E.$

As $P_T(u)=\{u\}$ for $u\in F(T)$ and if $u\notin F(T)=[0,2]$ then

$$P_T(u)=\left\{v\in T(u):\|u-v\|=d(u,Tu)=d\left(u,\left[0,\frac{u+2}{2}\right]\right)\right\}$$
$$=\left\{v\in T(u):\|u-v\|=\left\|u-\frac{u+2}{2}\right\|\right\}$$
$$=\left\{v\in T(u):u-v=\frac{u-2}{2}\right\} \text{ because } u>v\,\forall\, v\in T(u) \text{ where } u\in(0,1]$$
$$P_T(u)=\left\{v=\frac{u+2}{2}\right\}$$

Now, we show that P_T is nonexpansive mapping for all $u\in E.$ Let $u\in E$

$$H(P_T(u),P_T(p^*))=H\left(\frac{u+2}{2},p^*\right)=\left|\frac{u+2}{2}-p^*\right|\le|u-p^*|.$$

Above inequality establish that P_T is nonexpansive mapping for all $u\in E.$

As all the conditions of corollary 3.4 are satisfied, therefore there is atleast one fixed point of T. Now with the help of an iterative algorithm, we compute the fixed point of T. Taking $u_1=1\in[0,4]$ and $u_{n+1}=(1-\alpha_n)u_1+\alpha_n w_n$.

$$P_T(u_1)=\left\{\frac{1+2}{2}\right\}=\left\{1+\frac{1}{2}\right\} \text{ and } w_1\in P_T(u_1)\Rightarrow w_1=1+\frac{1}{2}$$

Then $u_2=(1-\alpha_1)u_1+\alpha_1 w_1$, so we have

$$u_2=\frac{1}{2}+\frac{1}{2}\cdot\frac{3}{2}=\frac{1}{2}+\frac{3}{4}<\frac{1}{2}+2$$
$$2-1<u_2<\frac{1}{2}+2$$

$$P_T(u_2)=\left\{\frac{1}{2}\left(\frac{5}{4}+2\right)\right\}=\left\{1+\frac{5}{8}\right\} \text{ and } w_2\in P_T(u_2)\Rightarrow w_2=1+\frac{5}{8}$$

Then $u_3=(1-\alpha_2)u_1+\alpha_2 w_2$, so we have

$$u_3=\frac{1}{3}+\frac{2}{3}\cdot\frac{13}{8}=\frac{1}{3}+\left(1+\frac{1}{12}\right)<\frac{1}{3}+2$$
$$\Rightarrow 2-\frac{2}{3}<u_3<\frac{1}{3}+2$$

$$P_T(u_3)=\left\{\frac{1}{2}\left(\frac{17}{12}+2\right)\right\}=\left\{1+\frac{17}{24}\right\} \text{ and } w_3\in P_T(u_3)\Rightarrow w_3=1+\frac{17}{24}$$

Then $u_4 = (1-\alpha_3)u_1 + \alpha_3 w_3,$ so we have

$$u_4 = \frac{1}{4} + \frac{3}{4}\cdot\frac{41}{24} = \frac{1}{4} + \left(1 + \frac{9}{32}\right) < \frac{1}{4} + 2$$

$$\Rightarrow 2 - \frac{1}{2} < u_4 < \frac{1}{4} + 2$$

$$P_T(u_4) = \left\{\frac{1}{2}\left(\frac{49}{32} + 2\right)\right\} = \left\{1 + \frac{49}{64}\right\} \text{and } w_4 \in P_T(u_4) \Rightarrow w_4 = 1 + \frac{49}{64}$$

Then $u_5 = (1-\alpha_4)u_1 + \alpha_4 w_4,$ so we have

$$u_5 = \frac{1}{5} + \frac{4}{5}\cdot\frac{113}{64} = \frac{1}{5} + \left(1 + \frac{53}{80}\right) < \frac{1}{5} + 2$$

$$\Rightarrow 2 - \frac{2}{5} < u_5 < \frac{1}{5} + 2$$

and so on

$$2 - \frac{2}{n} < u_n < \frac{1}{n} + 2, \ \forall n \in \mathbb{N}.$$

Therefore, $\lim_{n\to\infty} u_n = 2 \in [0,2].$

Conflicts of Interest: The authors declare that they have no conflicts of interest.

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