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Research Article

# Reserved Domination Number of graphs

G. Rajasekar[1] and G. Rajasekar[2]


## Abstract

In this paper the definitions of Reserved domination number is introduced as for the graph $G=(V,E)$ a subset $S$ of $V$ is called a Reserved Dominating Set $(RDS)$ of $G$ if(i) $\mu$ be any nonempty proper subset of $S$;(ii) Every vertex in $V-S$ is adjacent to a vertex in $S$. The dominating set $S$ is called a minimal reserved dominating set if no proper subset of $S$ containing $\mu$ is a dominating set. The set $\mu$ is called Reserved set.The minimum cardinality of a reserved dominating set of $G$ is called the reserved domination number of $G$ and is denoted by $R_{(k)}-\gamma(G)$ where $k$ isthe number of reserved vertices. Using these definitions the reserved domination number for Path graph $P_n$, Cycle graph $C_n$, Wheel graph $W_n$, Star graph $S_n$, Fan graph $F_{1,n}$, Helm graph $H_n$, Complete graph $K_n$, Complete Bipartite graph $K_{m,n}$, Antiprism graph $AP_n$ and Ladder rung graph $nP_2$ are found.


***Keywords:*** Dominating set, Domination Number, Reserved dominating set, Reserved domination number.

## 1. Introduction

Domination in graphs has wide applications to several fields such as mobile Tower installation, School Bus Routing, Computer Communication Networks, Radio Stations, Locating Radar Stations Problem, Nuclear Power Plants Problem, Modeling Biological Networks, Modeling Social Networks, Facility Location Problems, Coding Theory and Multiple Domination Problems with hierarchical overlay networks. Domination arises in facility location problems, where the number of facilities (e.g., Mobile towers, bus stop, primary health center, hospitals, schools, post office, hospitals, fire stations) is fixed and one attempts to minimize the distance that a person needs to travel to get to the nearest facility. A similar problem occurs when the maximum distance to a facility is fixed and government or any service provider attempts to minimize the number of facilities necessary so that everyone is serviced. Concepts from domination also appear in problems involving finding sets of representatives, in monitoring communication or electrical networks, and in land surveying.

Let $G=(V,E)$ be a graph. A subset $S$ of $V$ is called a dominating set [1] of $G$ if every vertex in $V-S$ is adjacent to a vertex in $S$. A dominating set $S$ is called a minimal dominating set if no proper subset of $S$ is a dominating set. The minimum cardinality of a dominating set of $G$ is called the domination number of $G$ and is denoted by $\gamma(G)$. The maximum cardinality of a

[1]Research Scholar, Department of Mathematics, Jawahar Science College, Neyveli, r.g.raja3007@gmail.com
[2]Associate Professor, Department of Mathematics, Jawahar Science College, Neyveli, grsmaths@gmail.com

minimal dominating set of $G$ is called the upper domination number of $G$ and is denoted by $\Gamma(G)$ .In 1984, M. S. Jacobson and L. F. Kinch [6] have developed an algorithm for the domination number of product of graphs. In the year 2014, Abdul Jalil M. Khalaf and Sahib ShayyalKahat[8] have developed domination polynomial of the domination in graphs.

Consider the situation of installing minimum number of Mobile phone towers so that it will be utilized by all the people living in towns or villages. Here each town or village is considered as separate entity called as vertex. These towns and villages are connected by roads(In this case Ariel distance) called edges. The situation of installing minimum number of Mobile phone Towers is **domination problem and this minimum number is domination number**.

In real life situation it is not always in practice of installing the Towers only at the public utility pattern. If the installing authorities are very much interested in installing the tower, nearer to their residence or nearer to the residence of VIPs or to their favorites' residence then the concerned person will reserve some of the installation points without any concern about the nearer or proximity of the other towns or villages. Such type installation points are a called reserved installation points. This situation motives the development of the reserved domination points. These reserved points areautomatically included in the domination set.

In this paper all the notations and terminologies used are the notations used by C. Berge[2], Harary[3], J. A. Bondy and U. S. R. Murty[4], T.W. Haynes, S.T. Hedetniemi, P.J. Slater[5], Stephen John.B and Krishna Girija.B[7], Ayhan A. khalil[9], and SirilukIntaja and ThaninSitthiwirattham [10].

## 2. Preliminaries results

**Definition 2.1.**Dominating Set [1]

Let $G=(V,E)$ be a graph. A subset $S$ of $V$ is called a dominating set of $G$ if every vertex in $V\setminus S$ is adjacent to a vertex in $S$. A dominating set $S$ is called a minimal dominating set if no proper subset of $S$ is a dominating set. The minimum cardinality of a dominating set of $G$ is called the domination number of $G$ and is denoted by $\gamma(G)$. The maximum cardinality of a minimal dominating set of $G$ is called the upper domination number of $G$ and is denoted by $\Gamma(G)$.

**Domination Number**

(i) Path Graph: $\gamma(P_n)=\left\lceil \frac{n}{3} \right\rceil$

(ii) Cycle Graph: $\gamma(C_n)=\left\lceil \frac{n}{3} \right\rceil$

(iii) Wheel Graph: $\gamma(W_n)=1$

(iv) Star Graph: $\gamma(S_n)=1$

(v) Helm Graph: $\gamma(H_n)=n$

(vi) Fan Graph: $\gamma(F_{1,n})=1$

(vii) Complete Graph: $\gamma(K_n)=1$

(viii) Complete Bipartite Graph: $\gamma(K_{m,n})=2$

(ix) Antiprism Graph: $\gamma\left(AP_n\right)=\left\lfloor \frac{2(n+2)}{5} \right\rfloor$

(x) Ladder Rung Graph: $\gamma\left(nP_2\right)=n$

## 3. Main Results

**Definition 3.1:** Reserved Domination set

Let $G=(V,E)$ be a graph. A subset $S$ of $V$ is called a Reserved Dominating Set $(RDS)$ of $G$ if

(i) $\mu$ be any nonempty proper subset of $S$.

(ii) Every vertex in $V-S$ is adjacent to a vertex in $S$.

The dominating set $S$ is called a minimal reserved dominating set if no proper subset of $S$ containing $\mu$ is a dominating set. The set $\mu$ is called Reserved set. The minimum cardinality of a reserved dominating set $S$ of $G$ is called the reserved domination number of $G$ and is denoted by $R-\gamma(G)$.

**Definition 3.2:** $k$-Reserved Domination set

Let $G=(V,E)$ be a graph. A subset $S$ of $V$ is called a Reserved Dominating Set $(RDS)$ of $G$ if

(i) $\mu$ be any nonempty proper subset of $S$.

(ii) Every vertex in $V-S$ is adjacent to a vertex in $S$.

The dominating set $S$ is called a minimal $k$-reserved dominating set if no proper subset of $S$ containing $\mu$ is a dominating set. The set $\mu$ is called $k$-Reserved set.

The minimum cardinality of a $k$-reserved dominating set $S$ of $G$ is called the $k$-reserved domination number of $G$ and is denoted by $R_{(k)}-\gamma(G)$ where $k$ is the number of reserved vertices.

### 3.1. Reserved domination number for graphs with $\gamma=1$

**Definition 3.1.1.** Wheel Graph

A wheel graph is a graph $W_n$ formed by connecting a single universal vertex to all vertices of a cycle. To denote a wheel graph with $n+1$ vertices $(n\geq 3)$, which is formed by connecting a single vertex to all vertices of a cycle of length $n$.

**Theorem 3.1.2.** For the wheel graph $W_n, n\geq 4$ the reserved domination number is

$$R_{(1)}-\gamma\left(W_n,\mu\right)=\begin{cases}1 & \text{if } \mu=u \\ 2 & \text{if } \mu=v_k\ (k=1,2,3,\ldots,n)\end{cases}$$

*Proof:*

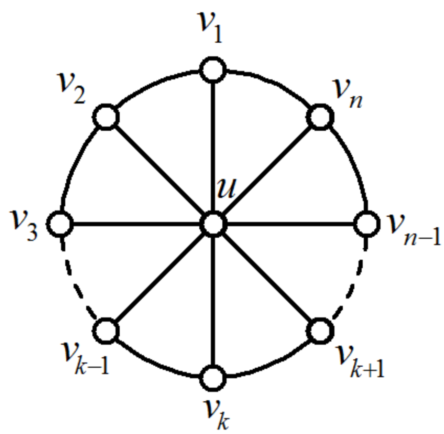


**Fig 1: A wheel graph** $W_n$

Let the vertex set of wheel graph $W_n$ be $\mu = \{u, v_1, v_2, v_3, ..., v_n\}$.

For $n = 3$, the reserved domination number is $R_{(1)} - \gamma(W_3) = 1$.

Case (i): $\mu = u$.

Let $u$ is reserved vertex. As $u$ must be chosen in dominating set and $u$ dominates all other vertices, the required dominating set is $\{u\}$.

Thus, $R_{(1)} - \gamma(W_n, u) = 1$.

Case (ii): $\mu = v_k \ (k = 1, 2, 3, ..., n)$.

Let $v_k$ is the reserved vertex. As $v_k$ must be chosen for constructing the required dominating set and $v_{k-1}$ and $v_{k+1}$ are only dominated. The remaining not dominated vertices are $\{v_1, v_2, ..., v_{k-2}, v_{k+2}, ..., v_n\}$.

To dominate the remaining vertices, choose the vertex $u$.

Hence the required dominating set is $\{v_k, u\}$.

Thus, $R_{(1)} - \gamma(W_n, \mu) = 2$, where $\mu = v_k, k = 1, 2, 3, ..., n$.

**Definition 3.1.3.** Star Graph

A star graph $S_n$ is the complete bipartite graph $K_{1,n}$, a tree with one internal node and $n$ leaves (but no internal nodes and $k+1$ leaves when $k \leq 1$).

**Theorem 3.1.4.** For the star graph $S_n, n \geq 2$ the reserved domination number is

$$R_{(1)} - \gamma(S_n, \mu) = \begin{cases} 1 & \text{if } \mu = u \\ 2 & \text{if } \mu = v_k \ (k = 1, 2, 3, ..., n) \end{cases}$$

*Proof:*

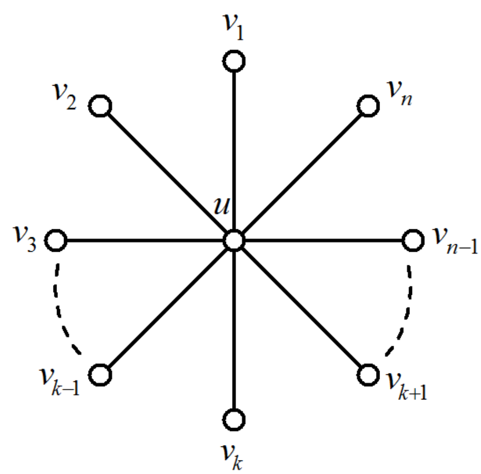


**Fig 2: A star graph $S_n$**

Let the vertex set of star graph $S_n$ be $\mu = \{u, v_1, v_2, v_3, ..., v_n\}$.

For $n = 1$, the reserved domination number is $R_{(1)} - \gamma(S_1) = 1$.

Case (i): $\mu = u$.

Let $u$ is reserved vertex. As $u$ must be chosen in dominating set and $u$ dominates all other vertices, the required dominating set is $\{u\}$.

Thus, $R_{(1)} - \gamma(S_n, u) = 1$.

Case (ii): $\mu = v_k\ (k = 1, 2, 3, ..., n)$.

Let $v_k$ is the reserved vertex. As $v_k$ must be chosen for constructing the required dominating set and $v_{k-1}$ and $v_{k+1}$ are only dominated. The remaining not dominated vertices are $\{v_1, v_2, ..., v_{k-2}, v_{k+2}, ..., v_n\}$.

To dominate the remaining vertices, choose the vertex $u$.

Hence the required dominating set is $\{v_k, u\}$.

Thus, $R_{(1)} - \gamma(S_n, \mu) = 2$, where $\mu = v_k, k = 1, 2, 3, ..., n$.

**Definition 3.1.5.** Fan graph

A fan graph $F_{1,n}$ is defined as the graph join $K_1 + P_n$, where $K_1$ is the singleton graph on 1 node and $P_n$ is the path graph on $n$ nodes.

**Theorem 3.1.6.** For the fan graph $F_{1,n}$, $n \geq 4$ the reserved domination number is $R_{(1)} - \gamma(F_{1,n}, \mu) = \begin{cases} 1, & \text{if } \mu = u \\ 2, & \text{if } \mu = v_k\ (k = 1, 2, 3, ..., n) \end{cases}$.

*Proof:*

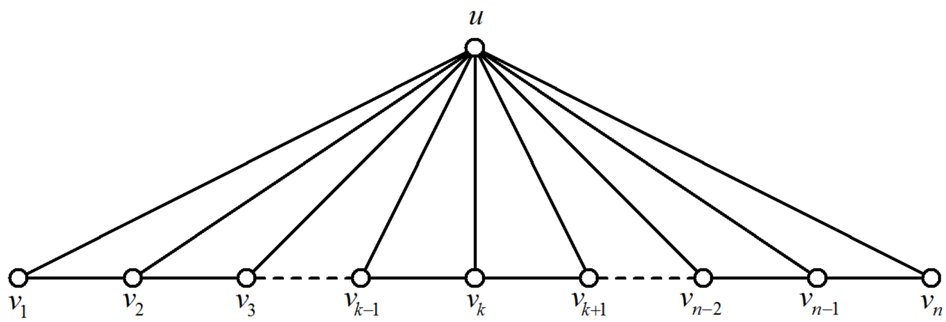


**Fig 3: A fan graph $F_{1,n}$**

Let the vertex set of fan graph $F_{1,n}$ be $\mu = \{u, v_1, v_2, v_3, ..., v_n\}$.

For $n=1$, the reserved domination number is $R_{(1)} - \gamma(F_{1,1}) = 1$.

For $n=2$, the reserved domination number is $R_{(1)} - \gamma(F_{1,2}) = 1$.

For $n=3$, the reserved domination number is $R_{(1)} - \gamma(F_{1,3}, \mu) = \begin{cases} 1 \text{ if } \mu = u, v_2 \\ 2 \text{ if } \mu = v_1, v_3 \end{cases}$.

Case (i): $\mu = u$.

Let $u$ is reserved vertex. As $u$ must be chosen in dominating set and $u$ dominates all other vertices, the required dominating set is $\{u\}$.

Thus, $R_{(1)} - \gamma(F_{1,n}, u) = 1$.

Case (ii): $\mu = v_k \ (k = 1, 2, 3, ..., n)$.

Let $v_k$ is the reserved vertex. As $v_k$ must be chosen for constructing the required dominating set and $v_{k-1}$ and $v_{k+1}$ are only dominated. The remaining not dominated vertices are $\{v_1, v_2, ..., v_{k-2}, v_{k+2}, ..., v_n\}$.

To dominate the remaining vertices, choose the vertex $u$.

Hence the required dominating set is $\{v_k, u\}$.

Thus, $R_{(1)} - \gamma(F_{1,n}, \mu) = 2$, where $\mu = v_k, k = 1, 2, 3, ..., n$.

**Remark 3.1.7.** For the complete graph $K_n$, $n \geq 3$ the reserved domination number is $R_{(1)} - \gamma(K_n) = 1$.

### 3.2. Reserved domination number for graphs with $\gamma = 2$

**Theorem 3.2.1.** For the bistargraph $B_{m,n}$, $n \geq 2$ the reserved domination number is

$$R_{(1)} - \gamma(B_{m,n}, \mu) = \begin{cases} 2, \text{ if } \mu = u, v \\ 3, \text{ if } \mu = u_k \ (k = 1, 2, 3, ..., m), v_k \ (k = 1, 2, 3, ..., n) \end{cases}.$$

*Proof:*

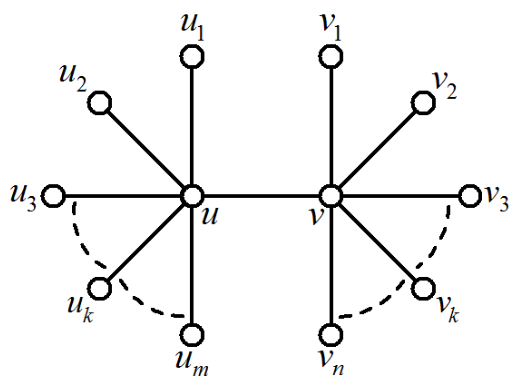


**Fig 4: A bistar graph** $B_{m,n}$

Let the vertex set of bistar graph $B_{m,n}$ be $\mu=\{V_1,V_2\}$ where $V_1=\{u,u_1,u_2,u_3,...,u_m\}$ and $V_2=\{v,v_1,v_2,v_3,...,v_n\}$.

Case (i): $\mu=u$.

Let $u$ is the reserved vertex. As $u$ must be chosen in dominating set, $u$ dominates the vertices $\{u_1,u_2,u_3,...,u_m\}\cup\{v\}$. The remaining vertices which are not dominated by $u$ are $\{v_1,v_2,...,v_n\}$.

To dominate the remaining vertices, choose the vertex $v$.

Hence the required dominating set is $\{u,v\}$.

Thus, $R_{(1)}-\gamma\left(B_{m,n},u\right)=2$.

Similarly, we can prove for the reserved vertex $\mu=v$.

Case (ii): $\mu=u_k\left(k=1,2,3,...,m\right)$.

Let $u_k$ is the reserved vertex. As $u_k$ must be chosen for constructing the required dominating set and $u$ only dominated. The remaining not dominated vertices are $\{u_1,u_2,u_3,...,u_{k-1},u_{k+1},...,u_m,v,v_1,v_2,v_3,...,v_n\}$.

To dominate the remaining vertices we must choose the vertices $u$ and $v$.

Hence the required reserved dominating set is $\{u_k,u,v\}$.

Thus, $R_{(1)}-\gamma\left(B_{m,n},\mu\right)=3$, where $\mu=u_k, k=1,2,3,...,m$.

Similarly, we can prove for the reserved vertex $\mu=v_k, k=1,2,3,...,n$.

**Remark3.2.2.**For the complete bipartite graph $K_{m,n}$, $m,n\geq 2$ the reserved domination number is $R_{(1)}-\gamma\left(K_{m,n}\right)=2$.

**3.3. Reserved domination Number of graphs with** $\gamma>2$

**Definition 3.3.1.**Path graph

The path graph $P_n$ is a tree with two nodes of vertex degree $1$, and the other $n-2$ nodes of vertex degree $2$.

**Theorem 3.3.2.**For the path graph $P_n$, the Reserved domination number is,

$R_{(1)}-\gamma(P_n,\mu)=1+\left\lceil\frac{k-2}{3}\right\rceil+\left\lceil\frac{n-(k+1)}{3}\right\rceil$ , if $\mu=v_k, k=1,2,3,...,n$.

*Proof:*

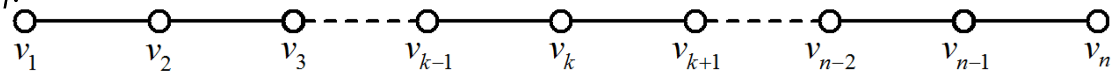


**Fig 5: A path graph $P_n$**

Let $P_n$ be a path graph with vertices $v_1,v_2,v_3,...,v_n$. Let $v_k$ is the reserved vertex.

Since $v_k$ must be chosen for constructing the required dominating set $v_{k-1}$ and $v_{k+1}$ are dominated. So now it is enough to find the domination number of the paths $P_{(1)}$ and $P_{(2)}$, where

$P_{(1)}=P_n[V_1]$ with $V_1=\{v_1,v_2,...,v_{k-2}\}$

$P_{(2)}=P_n[V_2]$ with $V_2=\{v_{k+2},v_{k+3},...,v_n\}$.

The length of the path $P_{(1)}=k-2$.

$$\gamma(P_{(1)})=\left\lceil\frac{k-2}{3}\right\rceil.$$

The length of the path $P_{(2)}=n-(k+1)$.

$$\gamma(P_{(2)})=\left\lceil\frac{n-(k+1)}{3}\right\rceil.$$

So now,

$$R_{(1)}-\gamma(P_n,\mu)=\gamma(P_{(1)})+|\{v_k\}|+\gamma(P_{(2)})$$
$$=\left\lceil\frac{k-2}{3}\right\rceil+1+\left\lceil\frac{n-(k+1)}{3}\right\rceil$$
$$=1+\left\lceil\frac{k-2}{3}\right\rceil+\left\lceil\frac{n-(k+1)}{3}\right\rceil, \text{ where } \mu=v_k, k=1,2,3,...,n.$$

**Definition 3.3.3.** Cycle graph

A cycle graph or circular graph is a graph that consists of a single cycle, or in other words, some number of vertices (at least 3, if the graph is simple) connected in a closed chain. The cycle graph with *n* vertices is called $C_n$. The number of vertices in $C_n$ equals the number of edges, and every vertex has degree 2; that is, every vertex has exactly two edges incident with it.

**Theorem 3.3.4.** For the cycle $C_n$, the reserved domination number is $R_{(1)}-\gamma(C_n,\mu)=\gamma(C_n)=\left\lceil\frac{n}{3}\right\rceil$ if $\mu=v_k, k=1,2,3,...,n$.

*Proof:*

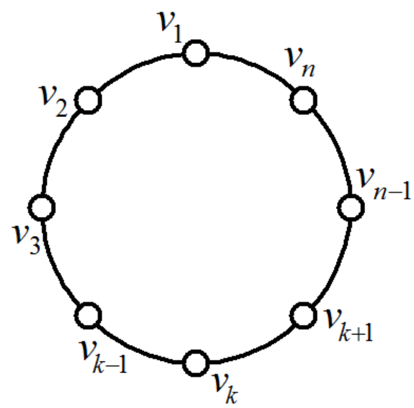


**Fig 6: A cycle graph $C_n$**

As cycle is a closed path, if one changes the reserved vertex to any point then making this same point as the starting point, there is no change in domination.

$$\therefore R_{(1)} - \gamma(C_n, \mu) = \gamma(C_n) = \left\lceil \frac{n}{3} \right\rceil.$$

**Definition 3.3.5.**Helm graph

The helm graph $H_n$ is the graph obtained from an $n-$wheel graph by adjoining a pendant edge at each node of the cycle.

**Theorem 3.3.6.**For the helm graph $H_n$, $n \geq 3$ the reserved domination number is

$$R_{(1)} - \gamma\left(H_n, \mu\right) = \begin{cases} n+1, & \text{if } \mu = u \\ n, & \text{if } \mu = v_k, w_k \left(k = 1, 2, 3, ..., n\right) \end{cases}.$$

*Proof:*

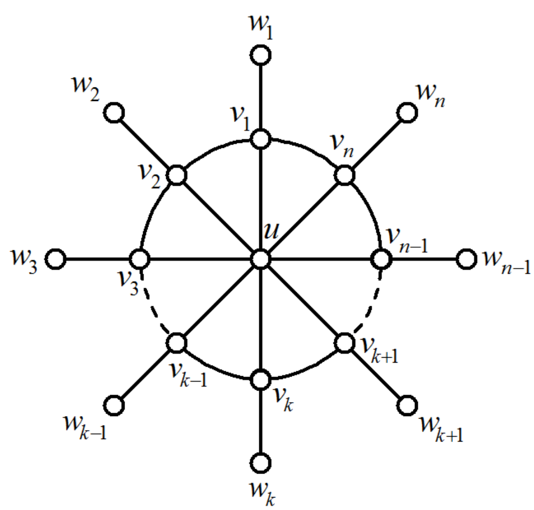


**Fig 7: A helm graph $H_n$**

Let the vertex set of the helm graph $H_n$ be $\mu = \{u, v_1, v_2, v_3, ..., v_n, w_1, w_2, w_3, ..., w_n\}$.

Case (i): Suppose $u$ is the reserved vertex.

Then $u$ must be in the dominating set and $u$ dominates the vertices $\{v_1, v_2, v_3, ..., v_n\}$. Now it is enough to find the dominating set for the remaining vertices $\{w_1, w_2, w_3, ..., w_n\}$.

The $H_n[V]$ with $V=\{w_1,w_2,w_3,...,w_n\}$ is nothing but $\overline{K_n}$.

Hence $R_{(1)}-\gamma(H_n,u)=1+\gamma(\overline{K_n})$

$=1+n$

$=n+1$.

Case (ii): Suppose $v_k, k=1,2,3,...,n$ is the reserved vertex.

Then $v_k$ must be in the dominating set and $v_k$ dominates the vertices $\{v_{k-1},v_{k+1}\}\cup\{u\}\cup\{w_k\}$. The remaining not dominated vertices are $\{v_1,v_2,...,v_{k-2},v_{k+2},...,v_n,w_1,w_2,...,w_{k-1},w_{k+1},...,w_n\}$.

To dominate the remaining vertices we must choose the vertices $\{v_1,v_2,...,v_{k-1},v_{k+1},...,v_n\}$ or $\{w_1,w_2,...,w_{k-1},w_{k+1},...,w_n\}$.

$$|\{v_1,v_2,...,v_{k-1},v_{k+1},...,v_n\}|=n-1$$

$$|\{w_1,w_2,...,w_{k-1},w_{k+1},...,w_n\}|=n-1.$$

Hence $R_{(1)}-\gamma(H_n,v_k)=1+n-1$

$=n$.

Case (iii): Suppose $w_k, k=1,2,3,...,n$ is the reserved vertex.

Then $w_k$ must be in the dominating set and $w_k$ dominates the vertex $v_k$. The remaining not dominated vertices $\{u,v_1,v_2,...,v_{k-1},v_{k+1},...,v_n,w_1,w_2,...,w_{k-1},w_{k+1},...,w_n\}$.

To dominate the remaining vertices, choose the vertices $\{v_1,v_2,...,v_{k-1},v_{k+1},...,v_n\}$.

Hence $R_{(1)}-\gamma(H_n,w_k)=|\{w_k\}|+|\{v_1,v_2,...,v_{k-1},v_{k+1},...,v_n\}|$

$=1+n-1$

$=n$.

**Remark 3.3.7.**For the antiprismgraph $AP_n$, $n\geq 3$ the reserved domination number is $R_{(1)}-\gamma(AP_n)=\left\lfloor \frac{2(n+2)}{5} \right\rfloor$.

**Remark 3.3.8.**For the ladder rung graph $nP_2$, the reserved domination number is $R_{(1)}-\gamma(nP_2)=n$.

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