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Resolving the Singularly Perturbed Two - Point Boundary Value Problem using Quadrature method

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Abstract:

In this research paper, we are fascinated to recount a Green shift to solve a specifically motivated to work two-point limit confidence problem with end limit layer in the stretch [0,1]. We investigated the appropriateness of the mathematical design while using the well-known Greens adjustment to a specific problem. During the approach, on two straight models, the quadrature method was applied. For exactness and to identify the error bound, the numerically computed results were compared to the analytical solutions. Computationally determined Findings were considered for the plan's assembly, and they are in appropriate agreement with the particular arrangement that is available in the literature. The analytical solutions found in the literature are closely associated with the simulation result.

Keywords:

Ordinary Differential Equation; Boundary Layer; Two-Point Boundary Problem, Quadrature method.

Introduction

Many physical problems from fluid mechanics, fluid dynamics, elasticity, magneto-hydrodynamics, plasma dynamics, oceanography, biological model, boundary layer theory,…etc are described mathematically by nonlinear integro-differential equations[1-3]. At present various techniques for numerical integration and methods for approximate solution of integro-differential equations and their applications to various physical models. Most of the nonlinear differential equations cannot be solved analytically. So it is required to obtain efficient numerical methods [4-7].Variable mesh methods for the solution of two point nonlinear boundary value problems; however, their methods are not applicable to differential equations with singular coefficients [8-12]. In recent years discussed a family of third order variable mesh methods for the solution of two point non-linear boundary value problems and obtained convergent solution for singular problems. In this context, we propose an efficient third order variable mesh method based on arithmetic average discretization for the solution of nonlinear integro-differential equation (4.1.1), which is applicable when the internal grid points of the solution region are both odd and even in number. Herewith we are resolving the singularly perturbed two - point boundary value problem using quadrature method depicted below.

Mathematical Details of the Discretization

$$v'' = F(y, v, v') + \int_0^1 K(y,t)dt, \quad 0 < y < 1, 0 < t < 1$$

$$v'' = F(y, v, v') + \int_0^1 K(y,t)dt, \quad 0 < y, t < 1 \tag{4.1.1}$$

$$u(0) = \delta_0, \quad u(1) = \delta_1 \tag{4.1.2}$$

δ_0, δ_1

$0 \le y, t \le 1$

$$v'' = H(y, v, v'), \; 0 < y < 1 \tag{4.1.3}$$

$v(y) \in C^6[0,1]$ and $v(y,t) \in C^4[0,1]$

$0 = y_0 < y_1 \cdots\cdots\cdots\cdots\cdots\cdots < y_{N+1} = 1. \quad m_{n+1} = y_{n+1} - y_n > 1.$

$n = 0(1)N + 1 y_i = y_0 + \sum_{n=1}^{i} m_n, \quad i = 1(1)N + 1.$

$\sigma_n = (m_{n+1}/m_n) > 0 \sigma_n = 1$

$y_{n+\frac{1}{2}} = y_n + \frac{\sigma_n m_n}{2}$ and $y_{n-\frac{1}{2}} = y_n - \frac{m_n}{2}$

$u(y) y_n \; U_n = v(y_n) \; v_n \; U_n$

$\int_0^1 \emptyset(y)dy \, \emptyset(y)$

$$\int_{y_{n-\frac{1}{2}}}^{y_{n+\frac{1}{2}}} \emptyset(y)dy = \alpha_{-1}\emptyset_{n-\frac{1}{2}} + \alpha_0 \emptyset_n + \alpha_1 \emptyset_{n+\frac{1}{2}} \tag{4.2.1}$$

$\alpha_{-1}, \alpha_0, \alpha_1 \; y_n \; \emptyset_n = \emptyset(y_n)$

$$y_{n+1/2} - y_n = \frac{m}{2} = y_n - y_{n-1/2}$$

$$\emptyset_{n+1/2} = \emptyset_n + \frac{m}{2}\emptyset'_n + \frac{m^2}{8}\emptyset''_n + \frac{m^3}{48}\emptyset'''_n + O(m^4) \tag{4.2.2a}$$

$$\int_{y_{n-\frac{1}{2}}}^{y_{n+\frac{1}{2}}} \emptyset(y)dy = \int_{y_{n-\frac{1}{2}}}^{y_{n+\frac{1}{2}}} \emptyset(y_n + (y - y_n))$$

$$\int_{y_{n-\frac{1}{2}}}^{y_{n+\frac{1}{2}}} \emptyset(y)dy = \int_{y_{n-\frac{1}{2}}}^{y_{n+\frac{1}{2}}} [\emptyset + (y - y_n)\emptyset'_n + \frac{(y-y_n)^2}{2}\emptyset''_n + \frac{(y-y_n)^3}{6}\emptyset'''_n + \frac{(y-y_n)^4}{24}\emptyset^{iv}_n + \cdots\cdots]dx$$

$$= [y\emptyset_n + \frac{(y-y_n)^2}{2}\emptyset_n' + \frac{(y-y_n)^3}{6}\emptyset_n'' + \frac{(y-y_n)^4}{24}\emptyset_n''' + \cdots\cdots]_{y_{n-\frac{1}{2}}}^{y_{n+\frac{1}{2}}}$$

$$= [m\emptyset_n + \frac{m^3}{24}\emptyset_n'' + \frac{m^5}{1920}\emptyset_n^{iv} + \cdots\cdots] \quad (4.2.3)$$

$\emptyset_n$

$\alpha_{-1} + \alpha_0 + \alpha_1 = m$

$-\frac{m}{2}\alpha_{-1} + \frac{m}{2}\alpha_1 = 0$

$-\alpha_{-1} + \alpha_1 = 0$

or $-\alpha_{-1} = \alpha_1$

$-\frac{m^2}{8}\alpha_{-1} + \frac{m^2}{8}\alpha_1 = \frac{m^3}{24}$

$\alpha_{-1} + \alpha_1 = \frac{m}{3}$

$\alpha_{-1} = \frac{m}{6}, \qquad \alpha_0 = \frac{4m}{6}, \qquad \alpha_1 = \frac{m}{6}$

$[y_n, y_{n+1}] \qquad y_n,\ y_{n+1/2}, y_{n+1}$

$$\int_{y_n}^{y_{n+1}} \emptyset(y)dy = \frac{m}{6}[\emptyset_n + 4\emptyset_{n+\frac{1}{2}} + \emptyset_{n+1}] \quad (4.2.4)$$

$[y_n, y_{n+1}]$ is m_{n+1}

$$\int_{y_n}^{y_{n+1}} \emptyset(y)dy = \frac{m_{n+1}}{6}[\emptyset_n + 4\emptyset_{n+\frac{1}{2}} + \emptyset_{n+1}], \quad n = 0,1,2,3 \ldots\ldots\ldots, N \quad (4.2.5)$$

$$\int_0^1 \emptyset(y)dy = \int_{x_0}^{x_1}\emptyset(y)dy + \int_{x_1}^{x_2}\emptyset(y)dy + \cdots\cdots\cdots\cdots + \int_{x_N}^{x_{N+1}}\emptyset(y)dy$$

$$= \sum_{n=0}^{N}\frac{m_{n+1}}{6}[\emptyset_n + 4\emptyset_{n+\frac{1}{2}} + \emptyset_{n+1}] \quad (4.2.6)$$

$V_n'' = H(y_n, V_n, V_n') = H_n$

$$\alpha_n^- = \left(\frac{dH}{dv}\right)_{y_n}, \beta_n = \left(\frac{dH}{dv'}\right)_{y_n} \quad (4.2.7)$$

$$\alpha_1 V_{n+1} + \alpha_0 V_n + \alpha_{-1}V_{n-1} = m_n^2[b_1 H_{n+\frac{1}{2}} + b_0 H_n + b_{-1}H_{n-\frac{1}{2}}] + T_n \quad (4.2.8)$$

Where $T_n = O(m_n^5)$

$$\alpha_{-1} + \alpha_0 + \alpha_1 = 0 \quad (4.2.9a)$$

$$\alpha_1\sigma_n m_n - \alpha_{-1}m_n = 0 \quad (4.2.9b)$$

$\alpha_1\sigma_n^2 m_n^2 + \alpha_{-1}m_n^2 = 2m_n^2(b_1 + b_0 + b_{-1})$

$$\alpha_1\sigma_n^2 + \alpha_{-1} = 2(b_1 + b_0 + b_{-1}) \qquad (4.2.9b)$$

$$\alpha_1\sigma_n^3 m_n^3 - \alpha_{-1}m_n^3 = 3m_n^3(b_1\sigma_n - b_{-1})$$
$$\alpha_1\sigma_n^3 - \alpha_{-1} = 3(b_1\sigma_n - b_{-1}) \qquad (4.2.9d)$$

$$\alpha_1\sigma_n^4 m_n^4 + \alpha_{-1}m_n^4 = 3m_n^4(b_1\sigma_n^2 + b_{-1})$$
$$\alpha_1\sigma_n^4 + \alpha_{-1} = 3(b_1\sigma_n^2 + b_{-1}) \qquad (4.2.9e)$$

$$3(b_1\sigma_n - b_{-1}) = \sigma_n^3 - \sigma_n \qquad (4.2.10a)$$
$$3(b_1\sigma_n^2 + b_{-1}) = \sigma_n^4 + \sigma_n \qquad (4.2.10b)$$

$$b_1 = \frac{\sigma_n^2}{3}$$

$O(m_n^3)$,

$$(V_{n+1} - (1+\sigma_n)V_n + \sigma_n V_{n-1}) = \frac{\sigma_n m_n^2}{3}\left[\sigma_n H_{n+\frac{1}{2}} + \frac{(1+\sigma_n)}{2}H_n + H_{n-\frac{1}{2}}\right] + O(m_n^5), \sigma_n \neq 1 \qquad (4.2.11)$$

$$\bar{V}_{n+\frac{1}{2}} = \alpha_1 V_{n+1} + \alpha_0 V_n$$
$$\bar{V}_{n+\frac{1}{2}} = \frac{1}{2}(V_{n+1} + V_n) = V_{n+\frac{1}{2}} + \frac{\sigma_n^2 m_n^2}{8}V_n'' + O(m_n^3), \qquad (4.2.12)$$

$$\bar{V}_{n-\frac{1}{2}} = \frac{1}{2}(V_{n-1} + V_n) = V_{n-\frac{1}{2}} + \frac{m_n^2}{8}V_n'' - O(m_n^3), \qquad (4.2.13)$$

$$\alpha_0 = -1, \qquad \alpha_1 = 1$$

$$\bar{V}'_{n+\frac{1}{2}} = \frac{1}{\sigma_n m_n}(V_{n+1} - V_n) = V'_{n+\frac{1}{2}} + \frac{\sigma_n^2 m_n^2}{24}V_n''' + O(m_n^3), \qquad (4.2.14)$$

$$\bar{V}'_{n-\frac{1}{2}} = \frac{1}{m_n}(V_n - V_{n-1}) = V'_{n-\frac{1}{2}} + \frac{m_n^2}{24}V_n''' - O(m_n^3), \qquad (4.2.15)$$

$$\alpha_1 + \alpha_0 + \alpha_{-1} = 0 \qquad (4.2.16a)$$
$$\alpha_1\sigma_n m_n - \alpha_{-1}m_n = 1 \qquad (4.2.16b)$$
$$\alpha_1\sigma_n^2 + \alpha_{-1} = 0 \qquad (4.2.16c)$$
$$\alpha_1\sigma_n m_n(1+\sigma_n) = 1$$

$$\alpha_1 = \frac{1}{\sigma_n m_n(1+\sigma_n)}$$
$$\alpha_{-1} = -\frac{\sigma_n^2}{\sigma_n m_n(1+\sigma_n)}$$
$$\alpha_0 = \frac{(1-\sigma_n^2)}{\sigma_n m_n(1+\sigma_n)}$$

$$\bar{H}_{n+\frac{1}{2}} = H_{n+\frac{1}{2}} + \frac{\sigma_n^2 m_n^2}{8} V_n'' \alpha_{n+\frac{1}{2}} + \frac{\sigma_n^2 m_n^2}{24} V_n''' \beta_{n+\frac{1}{2}} + O(m_n^3), \quad \sigma_n \neq 1$$

$$= H_{n+\frac{1}{2}} + \frac{\sigma_n^2 m_n^2}{24}(3V_n'' \alpha_n + V_n''' \beta_n) + O(m_n^3), \qquad \sigma_n \neq 1 \qquad (4.2.20a)$$

$$= H_{n-\frac{1}{2}} + \frac{m_n^2}{24}(3V_n'' \alpha_n + V_n''' \beta_n) - O(m_n^3), \ \sigma_n \neq 1 \qquad (4.2.20b)$$

$$\bar{H}_{n+\frac{1}{2}} + \bar{H}_{n-\frac{1}{2}} = 2V_n'' + \frac{m_n}{2}(\sigma_n - 1)V_n''' + \frac{m_n^2}{24}(1+\sigma_n^2)(3V_n''\alpha_n + V_n'''\beta_n + 3V_n'^{v}) + O(m_n^3), \quad \sigma_n \neq 1 \qquad (4.2.21a)$$

$$\bar{H}_{n+\frac{1}{2}} - \bar{H}_{n-\frac{1}{2}} = \frac{m_n}{2}(1+\sigma_n)V_n''' + \frac{m_n^2}{24}(\sigma_n^2 - 1)(3V_n''\alpha_n + V_n'''\beta_n + 3V_n'^{v}) + O(m_n^3), \sigma_n \neq 1 \qquad (4.2.21b)$$

$$\bar{\bar{V}}_n = V_n + \alpha m_n^2 (\bar{H}_{n+\frac{1}{2}} + \bar{H}_{n-\frac{1}{2}}) \qquad (4.2.22a)$$

$$\bar{\bar{V}}_n' = \bar{V}_n' + b m_n (\bar{H}_{n+\frac{1}{2}} - \bar{H}_{n-\frac{1}{2}}) \qquad (4.2.22b)$$

$$\bar{\bar{V}}_n = V_n + 2\alpha m_n^2 V_n'' + O(m_n^3), \qquad \sigma_n \neq 1 \qquad (4.2.23a)$$

$$\bar{\bar{V}}_n' = \bar{V}_n' + b m_n (\bar{H}_{n+\frac{1}{2}} - \bar{H}_{n-\frac{1}{2}})$$

$$\bar{\bar{V}}_n' = \bar{V}_n' + \frac{m_n^2}{6}[\sigma_n + 3b(1+\sigma_n)]V_n''' + O(m_n^3), \qquad \sigma_n \neq 1 \qquad (4.2.23b)$$

$$\bar{H}_n = H_n + 2\alpha m_n^2 V_n'' \alpha_n + \frac{m_n^2}{6}[\sigma_n + 3b(1+\sigma_n)]V_n''' \beta_n + O(m_n^3), \ \sigma_n \neq 1 \ (4.2.25)$$

$$(1+\sigma_n^3) + 8\alpha(1+\sigma_n) = 0$$
$$(1 - \sigma_n + \sigma_n^2) + 2\big(\sigma_n + 3b(1+\sigma_n)\big) = 0$$
$$(1 + \sigma_n + \sigma_n^2) + 6b(1+\sigma_n) = 0$$
$$\alpha = \frac{-(1-\sigma_n+\sigma_n^2)}{8}, \ \alpha = \frac{-(1+\sigma_n+\sigma_n^2)}{6(1+\sigma_n)}$$
$$\bar{T}_n = O(m_n^5), \ \sigma_n \neq 1$$

Observations and Conclusion

Using three variable mesh points, we have discussed a new numerical method of accuracy of $O(m_n^3)$, based on arithmetic average discretizations for the solution of the nonlinear two point boundary value problem shown in above derived equations. We had developed a third order variable mesh method based on Numerov type discretization in which is only applicable to the solution space having odd number of grid points. Although the proposed variable mesh method involves more algebra, but applicable to the solution space having both odd and even number of internal grid points. In addition, the proposed method is directly applicable to singular problems and we do not require any fictitious points near the boundaries to incorporate the singular point. The numerical results indicate that the proposed method is computationally slightly better than the method discussed and applicable to the solution space with all internal grid points. In this singularly perturbed two - point boundary value problem is solved and analyzed.

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