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Some Recent Advances in Prime Cordial Labeling of Graphs

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Abstract

This paper deals with the prime cordial labeling of graphs obtained by performing graph operations namely, corona product and extension of vertex, on some well-known graphs. Further, certain interesting conjectures and open problems concerning prime cordial labeling are also formulated.

Keywords: Prime cordial labeling, Corona product, Extension of vertex.

1. Introduction

In this article, all of the graphs are simple, finite, connected, and undirected. Burton [2] and Hararay [5] are the names of the books we refer for number theory and graph theory respectively. Gallian [4] is complex survey on the various graph labeling problems with considerable bibliography. For definitions and other related literature we refer to [1, 6-10]. Cahit [3] pioneered the idea of cordial labeling. We refer to prime cordial labeling and prime cordial graph as "pcl" and "pcg" respectively, throughout this article.

Definition 1.1. [7] A pcl of a graph H^*having node set V_H^* is a one-one, onto map $g^*: V_H^* \to \{1,2,3, \dots, |V_H^*|\}$so that each edge '$u_1v_1$' is assigned the label 1 when *GCD* (g^* (u_1), g^* (v_1)) = 1 and 0 if *GCD* ($g^*(u_1)$, g^* (v_1)) >1, then the modulus of difference between the count of edges having labels 0 and 1 is at the most 1 i.e;$|e_{g^*}(0) - e_{g^*}(1)| \leq 1$. A graph is considered a pcg if it allows a pcl.

Note. One can easily recall that in graph theory the terms node and vertex, node set and vertex set are interchangeable.

2. Main Results

In this section, we discuss certain advanced results on the pcl of graphs.

2.1 PCL in the Context of Corona Product of Graphs

In this section, we accomplish some results on prime cordial labeling in the context of corona product of graphs.

Definition 2.1.1. [1] If $H^*$is a graph of order *r*, then the corona product of H^* with another graph K^*, represented by $H^* \Theta K^*$ is a graph acquired by considering one copy of H^* and *r* copies of K^*thereby connecting the r^{th} node of $H^*$by an edge to each node in the r^{th} copy of K^*.

Theorem 2.1.2. Corona product of path P_m with $\underline{K_1}$ denoted by $P_m \odot \underline{K_1}$ permits a pcl.

Proof. Let $P_m$be the given path having node set $V^s(P_m) = \{p_1, p_2, \dots, p_m\}$ and edge set $E^s(P_m) = \{p_i p_{i+1} : 1 \leq i \leq m-1\}$. Let G^s represent the graph acquired by considering corona product of $P_m$with $\underline{K_1}$ having node set $V^s(G^s) = V^s(P_m) \cup \{p_1', p_2' \dots, p_m'\}$ and edge set $E^s(G^s) = E(P_m) \cup \{p_i p_i' : 1 \leq \text{i} \leq \text{m}\}$.

Clearly the cardinalities of node set and edge set of G^s are equal to $2m \text{ and } 2m-1$ respectively. Consider the map $g^s: V^s(G^s) \rightarrow \{1,2,\ldots,2m\}$defined by $g^s(p_1)=2$, $g^s(p_i) = g^s(p_{i-1})+2$; $2 \leq i \leq m$, and $g^s(p_i^{'}) = g^s(p_i)-1$; $1 \leq i \leq m$. Evidently, $e_{g^s}(0) = m-1$ and $e_{g^s}(1) = m$. Hence, G^s is pcg.

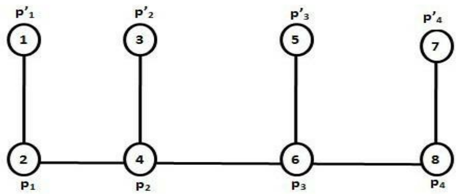

Figure 1. pcl of $P_4 \odot \underline{K_1}$

Theorem 2.1.3. Corona product of cycle C_p with $\underline{K_1}$ denoted by $C_p \odot \underline{K_1}$ allows a pcl.

Proof. Let C_p be the given cycle with node set $V^s(C_p) = \{c_1, c_2, \ldots, c_p\}$ and edge set $E^s(C_p) = \{c_i c_{i+1} : 1 \leq i \leq p-1\} \cup \{c_p c_1\}$. Let G^s represent the graph acquired by taking the corona product of C_p with $\underline{K_1}$ having node set $V^s(G^s) = V^s(C_p) \cup \{c_1^{'}, c_2^{'} \ldots, c_p^{'}\}$ and edge set $E^s(G^s) = E^s(C_p) \cup \{c_i c_i^{'} : 1 \leq i \leq p\}$. Clearly, $|V^s(G^s)| = 2p$ and $|E^s(G^s)| = 2p$. Consider the map $g^s: V^s(G^s) \rightarrow \{1,2,\ldots,2p\}$ given by $g^s(c_1) = 2$, $g^s(c_i) = g^s(c_{i-1}) + 2$; $2 \leq i \leq p$, and $g^s(c_i^{'}) = g^s(c_i) - 1$; $1 \leq i \leq p$. Following the above pattern, it is easy to note that $e_{g^s}(0) = p = e_{g^s}(1)$ which proves that G^s is a pcg.

*Theorem 2.1.4***.** Corona product of wheel graph with $\underline{K_1}$ allows a pcl.

Proof. Let W_n^α be the wheel graph having node set $V(W_n^\alpha) = \{w_0^\alpha\} \cup \{w_1^\alpha, w_2^\alpha, \ldots, w_n^\alpha\}$ and edge set $E(W_n^\alpha) = \{w_i^\alpha w_{i+1}^\alpha : 1 \leq i \leq n-1\} \cup \{w_n^\alpha w_1^\alpha\} \cup \{w_0^\alpha w_i^\alpha : 1 \leq i \leq n\}$. Let H^α represents the graph acquired by operating the corona product operation to get $W_n^\alpha \odot \underline{K_1}$ having $V(H^\alpha) = V(W_n^\alpha) \cup \{w_0^{'}, w_1^{'}, w_2^{'}, \ldots, w_n^{'}\}$ and edge set $E(H^\alpha) = E(W_n^\alpha) \cup \{w_i^\alpha w_i^{'}: 1 \leq i \leq n\} \cup \{w_0^\alpha w_0^{'}\}$. Clearly the cardinalities of node and edge set of H^α are equal to $2n+2 \text{ and } 3n+1$ respectively. Consider the map $g^*: V(H^\alpha) \rightarrow \{1,2,\ldots,2n+2\}$ defined under the two possibilities.

Case (i). When n is divisible by 2.

Fix $g^*(w_0^\alpha) = 2$, $g^*(w_1^\alpha) = 4$, $g^*(w_0^{'}) = 1$, $g^*(w_{\frac{n}{2}}^\alpha) = 6$, $g^*(w_{\frac{n}{2}+1}^\alpha) = 3$, $g^*(w_i^\alpha) = g^*(w_{i-1}^\alpha) + 4$, for $\frac{n}{2} + 2 \leq i \leq n$, $g^*(w_i^{'}) = g^*(w_i^\alpha) + 2$; $\frac{n}{2} + 1 \leq i \leq n$ and w_i^α and $w_j^{'}$ may be given the even labels, where $2 < i < \frac{n}{2}$ and $1 \leq j \leq \frac{n}{2}$.

Case (ii). When n is not divisible by 2.

Fix $g^*(w_0^\alpha) = 2$, $g^*(w_1^\alpha) = 4$, $g^*(w_0^{'}) = 2n+1$, $g^*(w_{\frac{n+1}{2}}^\alpha) = 6$, $g^*(w_{\frac{n+1}{2}}^{'}) = 1$, $g^*(w_{\frac{n+1}{2}+1}^\alpha) = 3$, $g^*(w_i^\alpha) = g^*(w_{i-1}^\alpha)+4$; $\frac{n+1}{2} + 2 \leq i \leq n$, and $g^*(w_i^{'}) = g^*(w_i^\alpha) + 2$; $\frac{n+1}{2} + 1 \leq i \leq n$. Allot the remaining even labels out of $\{1, 2, \ldots, 2n+2\}$ to w_i^α and $w_j^{'}$, where $1 < i < \frac{n+1}{2}$ and $1 \leq j < \frac{n+1}{2}$ in any pattern. Following the above patterns it is noteworthy that $|e_{g^*}(0) - e_{g^*}(1)| \leq 1$ and therefore H^αisa pcg.

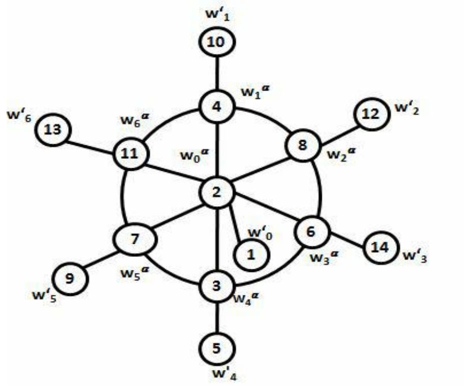

Figure 2. pcl of $W_6 O\ \underline{K_1}$

Theorem 2.1.5. Corona product of gear graph G_n with $\underline{K_1}$ is pcg.

Proof. Let G_n be the gear graph having node set $V(G_n)$ $=\{g_0\} \cup \{g_1, g_2, \dots, g_{2n}\}$ and edge set $E(G_n)$ = $\{g_i g_{i+1} : 1 \le i \le 2n\} \cup (g_{2n} g_1\} \cup \{g_0 g_i : i \in \{1,3,5 \dots, 2n-1\}$. Let H^* be the name given to corona product of G_n with $\underline{K_1}$ having node set $V(H^*) = V(G_n) \cup \{g_0^{'}, g_1^{'}, g_2^{'} \dots, g_{2n}^{'}\}$and edge set $E(H^*) = E(G_n) \cup \{g_i g_i^{'} : 1 \le i \le 2n\} \cup \{g_0 g_0^{'}\}$. Clearly the cardinalities of node and edge set of H^* are equal to 4n + 2 and 5n + 1 respectively. Consider the map $h^*: V(H^*) \to \{1,2, \dots, 4n+2\}$and let us perform the labeling as under.

Fix$h^*(g_0) = 2$, $h^*(g_0^{'}) = 1, h^*(g_1) = 4, h^*(g_n) = 6, h^*(g_{n+1}) = 3, h^*(g_i) = h^*(g_{i-1}) + 4; n+2 \le i \le 2n$. Next, $h^*(g_{n+1}^{'}) = 9, h^*(g_{n+2}^{'}) = 5$ and $h^*(g_i^{'}) = h^*(g_i)$ + 2; $n+3 \le i \le 2n$. Further, g_i and $g_j^{'}$should be marked with unutilized even labels, where 2 ≤ i< n, 1 ≤ j ≤ n. Following the above pattern it is easy to note that $|e_{h^*}(0) - e_{h^*}(1)| \le 1$ which shows that $H^*$is pcg.

*Theorem 2.1.6***.** Corona product of a flower graph Fl_n with $\underline{K_1}$ denoted by $Fl_n \odot \underline{K_1}$ permits a pcl.

Proof. Let Fl_nbe the flower graph having node set $V(Fl_n)$ = $\{v_0\} \cup \{v_1, v_2, \dots, v_n\} \cup \{u_1, u_2, \dots, u_n\}$ and edge set $E(Fl_n)$ $=\{v_0 v_i, v_0 u_i, v_i u_i : 1 \le i \le n\} \cup \{v_i v_{i+1} : 1 \le i \le n-1\} \cup \{v_n v_1\}$. Let H^* represents the graph acquired by considering the corona product of Fl_n with $\underline{K_1}$ having node set $V(H^*) = V(Fl_n) \cup \{u_0, v_i^{'}, u_i^{'} : 1 \le$ ***i≤n, 1≤j ≤n***and edge set ***EH∗=E(Fln)∪v0u0∪vivi′:1≤i≤n∪uiui′:1≤i≤n.*** Note that the cardinalities of node and edge set of $H^*$are equal to 4n+2 and $6n+1$ respectively. Consider the map $g^* : V(H^*) \to \{1, 2, \dots, 4n+2\}$ given by fixing $g^*(v_0) = 2, g^*(u_0)= 1, g^*(v_1)= 4, g^*(v_i) = g^*(v_{i-1}) + 4;2 \le$ i $\le$ n, $g^*(v_i^{'}) = g^*(v_i) + 2 ; 1 \le i \le n, g^*(u_i) = g^*(v_i) + 1;$ $1 \le i \le n$ and $g^*(u_i^{'}) = g^*(v_i) - 1;$ $1 \le i \le n$. Following this pattern, we accomplish that $|e_{g^*}(0) - e_{g^*}(1)| \le 1$. Thus H^* is pcg.

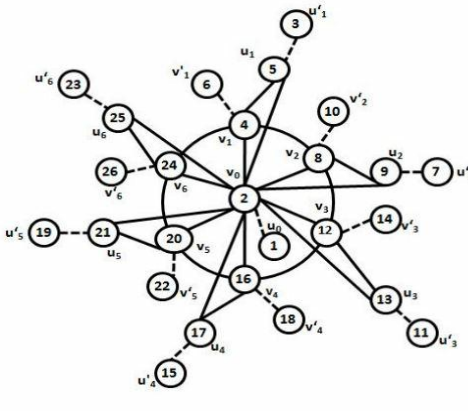

Figure 3. pcl of $Fl_6 \odot \underline{K_1}$

Theorem 2.1.7. Corona product of a path P_n with $\underline{K_2}$ permits a pcl.

Proof. Let $V(P_n) = \{p_1, p_2, \ldots, p_n\}$ and $E(P_n) = \{p_i p_{i+1} : 1 \le i \le n-1\}$ stands respectively for the set of nodes and edges of P_n . Let H^* represent the graph formed by taking the corona product of P_n with $\underline{K_2}$ having node set $V(H^*) = V(P_n) \cup \{p_1', p_2' \ldots, p_n'\} \cup \{p_1'', p_2'', \ldots, p_n''\}$ and edge set $E(H^*) = E(P_n) \cup \{p_i p_i' : 1 \le i \le n\} \cup \{p_i p_i'' : 1 \le i \le n\}$. Note that the cardinalities of node and edge set of P_n are equal to $3n$ and $3n-1$ respectively. Consider the map $g^*: V(H^*) \to \{1,2,\ldots,3n\}$ defined by two possibilities.

Case (i). When n is even.

Fix $g^*(p_1) = 2$, $g^*\left(p_{\frac{n}{2}+1}\right) = 1$, $g^*\left(p'_{\frac{n}{2}+1}\right) = 3$, $g^*\left(p''_{\frac{n}{2}+1}\right) = 5$, $g^*(p_i) = g^*(p_{i-1}) + 2; 2 \le i \le \frac{n}{2}$, $g^*\left(p_{\frac{n}{2}+1+i}\right) = g^*\left(p_{\frac{n}{2}+1+(i-1)}\right) + 6$; $1 \le i \le \frac{n}{2} - 1$, $g^*\left(p'_{\frac{n}{2}+1+i}\right) = g^*\left(p'_{\frac{n}{2}+1+(i-1)}\right) + 6$; $1 \le i \le \frac{n}{2} - 1$ and $g^*\left(p''_{\frac{n}{2}+1+i}\right) = g^*\left(p''_{\frac{n}{2}+1+(i-1)}\right) + 6$; $1 \le i \le \frac{n}{2} - 1$. Allocate the unutilized even labels to p_i', p_i'' for $1 \le i \le \frac{n}{2}$. Following the above pattern, we find that $|e_{g^*}(0) - e_{g^*}(1)| \le 1$.

Case (ii). When n is odd.

Fix $g^*(p_1) = 2$, $g^*\left(p_{\frac{n+1}{2}}\right) = 6$, $g^*\left(p'_{\frac{n+1}{2}}\right) = 3$, $g^*\left(p''_{\frac{n+1}{2}}\right) = 1$, $g^*\left(p_{\frac{n+1}{2}+1}\right) = 7$, $g^*\left(p_{\frac{n+1}{2}+1+i}\right) = g^*\left(p_{\frac{n+1}{2}+1+(i-1)}\right) + 6$; $1 \le i \le \frac{n-3}{2}$, $g^*\left(p'_{\frac{n+1}{2}+i}\right) = g^*\left(p_{\frac{n+1}{2}+i}\right) + 2$; $1 \le i \le \frac{n-1}{2}$ and $g^*\left(p''_{\frac{n+1}{2}+i}\right) = g^*\left(p_{\frac{n+1}{2}+i}\right) - 2$; $1 \le i \le \frac{n-1}{2}$. Allocate the unutilized even labels out of {1, 2,…,3n} to p_i, p_j' and p_k'' in any fashion, where $2 \le i \le \frac{n-1}{2}$, $1 \le j \le \frac{n-1}{2}$ and $1 \le k \le \frac{n-1}{2}$. Note that $e_{g^*}(0) = e_{g^*}(1) = \frac{3n-1}{2}$ which implies that $|e_{g^*}(0) - e_{g^*}(1)| \le 1$. Hence H^* is pcg.

Theorem 2.1.8. Corona product of cycle C_n with $\underline{K_2}$ denoted by $C_n \odot \underline{K_2}$ permits a pcl.

Proof. Let C_n be the given cycle graph having node set $V_c(C_n) = \{c_1, c_2, \ldots, c_n\}$ and edge set $E_c(C_n) = \{c_i c_{i+1} : 1 \le i \le n-1\} \cup \{c_n c_1\}$. Let H^* be the graph acquired by taking the corona product of C_n with $\underline{K_2}$ having $V_c(H^*) = V_c(C_n) \cup \{c_1', c_2' \ldots, c_n'\} \cup \{c_1'', c_2'', \ldots, c_n''\}$ as node set and $E_c(H^*) = E_c(C_n) \cup \{c_i c_i' : 1 \le i \le n\} \cup \{c_i c_i'' : 1 \le i \le n\}$ as an edge set. Note that the cardinalities of node and edge set of H^* are equal to $3n$ and $3n$ respectively. Consider the map $g^*: V_c(H^*) \to \{1,2,\ldots,3n\}$ defined under two possibilities.

Case (i).When n is even.

Fix $g^*(c_1) = 2$, $g^*(c_{\frac{n}{2}})$=6, $g^*\left(c_{\frac{n}{2}+1}\right) = 3$, $g^*\left(c_{\frac{n}{2}+2}\right) = 7$, $g^*\left(c_{\frac{n}{2}+2+i}\right) = g^*\left(c_{\frac{n}{2}+2+(i-1)}\right) + 6$; $1 \le i \le \frac{n}{2} - 2$, $g^*\left(c'_{\frac{n}{2}+1}\right) = 1$, $g^*\left(c'_{\frac{n}{2}+2}\right) = 9$, $g^*\left(c''_{\frac{n}{2}+1}\right) = 5$, $g^*\left(c'_{\frac{n}{2}+2+i}\right) = g^*\left(c'_{\frac{n}{2}+2+(i-1)}\right) + 6$; $1 \le i \le \frac{n}{2} - 2$ and $g^*\left(c''_{\frac{n}{2}+1+i}\right) = g^*\left(c''_{\frac{n}{2}+1+(i-1)}\right) + 6$; $1 \le i \le \frac{n}{2} - 1$. Allot the unutilized even labels out of the remaining labels to c_i, c_j', c_k'', where $2 \le i \le \frac{n}{2} - 1$, $1 \le j \le \frac{n}{2}$ and $1 \le k \le \frac{n}{2}$, in any fashion. Following this pattern it follows that $|e_{g^*}(0) - e_{g^*}(1)| \le 1$.

Case (ii). When n is odd.

Follow the labeling pattern of case (ii) of previous theorem, we observe that H^* allows a pcl.

Theorem 2.1.9. Corona product of P_n with $\underline{K_n}$ denoted by $P_n \odot \underline{K_n}$ permits a pcl.

Proof. Let P_n be path having $V(P_n) = \{p_1, p_2, \dots, p_n\}$ and $E(P_n) = \{p_i p_{i+1} : 1 \le i \le n-1\}$ as node and edge set respectively. Let H^* be the graph formed by taking the corona product of P_n with $\underline{K_n}$ having node set $V(H^*) = V(P_n) \cup \{k_{ij} : 1 \le i \le n, 1 \le j \le n\}$ and edge set $E(H^*) = E(P_n) \cup \{p_i k_{ij} : 1 \le \text{i} \le \text{n}, 1 \le j \le n\}$. Clearly the cardinalities of node and edge set of H^* are equal to $n^2 + n$ and $n^2 + n - 1$ respectively. Consider the map $g^*: V(H^*) \to \{1,2,\dots,n^2+n\}$ given by two possibilities.

Case (i). When n is even.

Fix $g^*(p_1) = 2, g^*(p_i) = g^*(p_{i-1}) + 2;\ 2 \le i \le \frac{n}{2},\ g^*(p_{\frac{n}{2}+1}) = 1$. Consider the sequence of consecutive prime numbers, say $q_{\frac{n}{2}+2}, q_{\frac{n}{2}+3}, \dots, q_n$ such that $n^2 + n \ge q_n > q_{n-1} > \cdots > q_{\frac{n}{2}+2}$. Fix $g^*(p_i) = q_i;\ \frac{n}{2} + 2 \le i \le n$. Allot the unutilized even labels to k_{ij} such that $1 \le i \le \frac{n}{2}, 1 \le j \le n$. Next assign unutilized odd labels to k_{ij} for $\frac{n}{2} + 1 \le i \le n, 1 \le j \le n$ simultaneously from $\{1,2,\dots,n^2+n\}$.

Case (ii). When n is odd.

Fix $g^*(p_1) = 2, g^*(p_i) = g^*(p_{i-1}) + 2;\ 2 \le i \le \lfloor\frac{n}{2}\rfloor,\ g^*\left(p_{\lceil\frac{n}{2}\rceil+1}\right) = 1$. Consider the sequence of consecutive prime numbers, say $q_{\lceil\frac{n}{2}\rceil+2}, q_{\lceil\frac{n}{2}\rceil+3}, \dots, q_n$ such that $n^2 + n \ge q_n > q_{n-1} > \cdots > q_{\lceil\frac{n}{2}\rceil+2}$. Fix $g^*(p_i) = q_i;\ \lceil\frac{n}{2}\rceil + 2 \le i \le n$. Allot the unutilized even labels to k_{ij} such that $1 \le i \le \lfloor\frac{n}{2}\rfloor, 1 \le j \le n$ and to $k_{\lceil\frac{n}{2}\rceil j}, 1 \le j \le \lfloor\frac{n}{2}\rfloor$. Allot the unutilized odd labels to $k_{\lceil\frac{n}{2}\rceil j}$ for $\lceil\frac{n}{2}\rceil \le j \le n$ and to k_{ij} for $\lceil\frac{n}{2}\rceil + 1 \le i \le n,\ 1 \le j \le n$ simultaneously from $\{1,2,\dots,n^2+n\}$. Following above pattern one can easily find that $\left|e_{g^*}(0) - e_{g^*}(1)\right| \le 1$, which prove that H^* is pcg.

Remark 2.1.10. One can easily derive pcl of the corona product of cycle C_n with $\underline{K_n}$ in similar lines with the above theorem.

2.2 PCL in the Context of Extension of Vertex

In this part, we discuss some results on pcl of graphs in the context of extension of node/nodes.

Definition 2.2.1. [7] Duplicating a node, say u_i^{α}, in G^u is accomplished by adding a new node u_i' so that $\text{N}(u_i') = \text{N}(u_i^{\alpha})$.

Definition 2.2.2. [7] Extension of a node, say u_i^{α}, is achieved by inserting a new node u_i' so that $\text{N}(u_i') = \text{N}[u_i^{\alpha}]$.

Theorem 2.2.3. Duplicating each node of a path graph P_n results in a pcg.

Proof: Let $u_1, u_2, \dots, u_n$ designates the set of nodes for path graph P_n. Let H^* be a graph formed by duplicating each node of P_n having node set V(H^*) = V(P_n) ∪{ v_i: 1≤ i ≤ n} and edge set E(H^*) = E(P_n) ∪{$u_{i-1}v_i$: 2 ≤ i ≤ n}∪{$v_i u_{i+1}$: 1≤ i ≤ n-1}. Clearly |V(H^*)| = 2n and |E(H^*)| = 3n-3. Consider a map g^*: V(H^*)→{ 1,2,...,2n} defined by two possibilities.

Case (i). When n is even.

Fix $g^*(u_{\frac{n}{2}}) = 6$, $g^*(v_{\frac{n}{2}}) = 2$, $g^*(u_{\frac{n}{2}+1}) = 5$, $g^*(v_{\frac{n}{2}+1}) = 3$. Allot the available even labels to u_i and v_i for $1 \le i < \frac{n}{2}$ in any fashion. Next, fix $g^*(u_i) = g^*(u_{i-1}) + 6$; i $> \frac{n}{2} + 1$, $g^*(v_i) = g^*(v_{i-1}) + 6$; i $> \frac{n}{2} + 1$. Let us assume that u_k and v_l be the largest indexed node, labeled by using above pattern. Label u_{k+1} with the largest unused odd label out of {1,2,...,2n}. Let $g^*(u_i) = g^*(u_{i-1}) - 12$ for k+2 ≤ i ≤ n, and for unlabeled nodes v_i, fix $g^*(v_i) = g^*(u_i) - 6$ for k+1 ≤ i ≤ n. Once this pattern ends, assign the unutilized label, if any, to the unlabeled node/nodes.

Case (ii). When n is odd.

Fix $g^*(u_{\lceil\frac{n}{2}\rceil})$= 6, $g^*(v_{\lceil\frac{n}{2}\rceil})$ = 2, $g^*(u_{\lceil\frac{n}{2}\rceil+1})$ = 5, $g^*(v_{\lceil\frac{n}{2}\rceil+1})$ = 3. Allot the remaining even labels to u_i; $1 \leq$ i <$\lceil\frac{n}{2}\rceil$ and to v_j ; 2 ≤ j <$\lceil\frac{n}{2}\rceil$ in any fashion. Next, fix $g^*(u_i) = g^*(u_{i-1})$ + 6; i >$\frac{n}{2}+1$, $g^*(v_i) = g^*(v_{i-1})$ + 6; i > $\frac{n}{2}+1$. Let us assume that u_k be the largest indexed node labeled with above pattern. Assign the largest unused label out of {1, 2,..., 2n} to u_{k+1}. Assign $g^*(u_i) = g^*(u_{i-1})$ - 12 for i≥ k+2 and for unlabeled nodes v_i, fix $g^*(v_i) = g^*(u_i)$ – 6 for i ≥ k+1. Once this pattern ends, allot the unutilized labels to the remaining unmarked node/nodes.

In view of the above cases, it follows that $|e_{g^*}(0) - e_{g^*}(1)| \leq 1$. Hence, H^* is pcg.

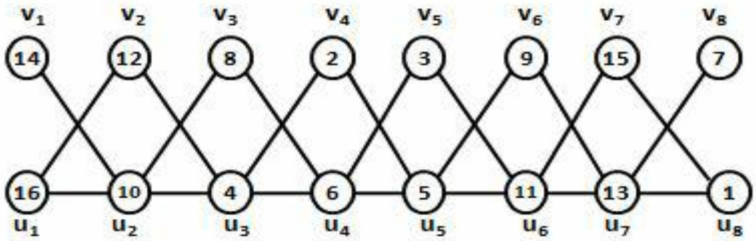

Figure 4. pcl of graph formed by duplication of each node in P_5

Remark 2.2.4. It is easy to deduce the pcl of the graph formed by duplicating each node of cycle graph and wheel graph on similar lines with the above theorem.

Theorem 2.2.5. Extension of an arbitrary node of a cycle graph C_n results in a pcg for n ≥ 7.

Proof. Let $c_1, c_2, \ldots, c_n$ designates the nodes ofC_n. Let H^* be a graph produced by taking the extension of an arbitrary node of C_n. Let us consider the extension of node c_1 and let w be the freshly inserted node. Clearly cardinalities of node and edge set of $H^*$are equal to n+1 and n+3 respectively. Consider a vertex labeling g^*: V(H^*) → { 1, 2 ,...,n+1} as given. Let $g^*(c_1)$= 6, $g^*(c_n)$ = 2 and $g^*(w)$= 4. Now we have the under mentioned possibilities.

Case (i). When n is even.

$g^*(c_i)= g^*(c_{i-1})+ 2$; $2 \leq i \leq \frac{n}{2} - 2$, $g^*\left(c_{\frac{n}{2}-1}\right)$ = 1, $g^*(c_i)= g^*(c_{i-1})$ +2; $\frac{n}{2} \leq$ i ≤ n-1.

Case (ii). When n is odd.

$g^*(c_i)= g^*(c_{i-1})+ 2$; $2 \leq i \leq \frac{n-1}{2} - 1$, $g^*\left(c_{\frac{n-1}{2}}\right)$ = 1, $g^*(c_i)= g^*(c_{i-1})$ +2; $\frac{n-1}{2}+1 \leq$ i ≤ n-1.

Note that $|e_{g^*}(0) - e_{g^*}(1)| \leq 1$ for both the cases which ensures that H^* is pcg.

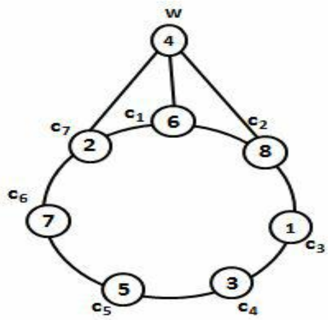

Figure 5. pcl of extension of c_1 in C_6

Theorem 2.2.6. The graph acquired by performing the extension of a rim vertex of wheel W_n chosen at random allows a pcl for n > 7.

Proof. Let V (W_n) = $\{w_0, w_1, w_2 \dots, w_n\}$ be the node set of W_n in which w_0 is central node and rest are rim nodes. Let H^* be a graph produced by taking extension of an arbitrary rim node. Let us take the extension of w_1 and let x be the freshly inserted node. Clearly the cardinalities of node and edge sets of H^* are equal to n+ 2 and 2n+ 4 respectively. Consider a map g^*: V(H^*) → { 1,2,..., n+2} defined under three conditions.

Case (i). When n is even.

$g^*(w_0)$= 2, $g^*(x) = 6$, $g^*(w_1) = 8$, $g^*(w_n) = 4$, $g^*\left(w_{\frac{n}{2}-1}\right) = 1$, $g^*\left(w_{\frac{n}{2}}\right) = 3$, $g^*\left(w_{\frac{n}{2}+1}\right) = 9$, $g^*(w_i)$= $g^*(w_{i-1})$ +2; 2 $\leq i \leq \frac{n}{2} - 2$. Allocate the unutilized labels to unmarked nodes namely $w_{\frac{n}{2}+2}, w_{\frac{n}{2}+3}, \dots, w_{n-1}$ simultaneously from {1,2,..., n+2}.

Case (ii). When n is odd and $n \equiv 2(mod\ 3)$.

$g^*(w_0)$= 2, $g^*(x) = 6$, $g^*(w_1) = 8$, $g^*(w_n) = 4$, $g^*(w_{n-1}) = 1$,$g^*\left(w_{\frac{n+1}{2}-2}\right) = 3$, $g^*\left(w_{\frac{n+1}{2}-1}\right)$ = 9, and $g^*(w_i)$ = $g^*(w_{i-1})$+2; 2≤ $i \leq \frac{n+1}{2} - 3$. Allocate unutilized labels to unlabeled nodes namely $w_{\frac{n+1}{2}}, w_{\frac{n+1}{2}+1}, \dots,$ w_{n-2} simultaneously from {1,2,..., n+2}.

Case (iii). When n is odd and $n \not\equiv 2(mod\ 3)$.

$g^*(w_0)$= 2, $g^*(x) = 8$, $g^*(w_1) = 10$, $g^*(w_n) = 4$, $g^*\left(w_{\frac{n+1}{2}-3}\right) = 6$, $g^*\left(w_{\frac{n+1}{2}-2}\right)$ = 3, $g^*\left(w_{\frac{n+1}{2}-1}\right) = 9$ and $g^*(w_i)$=$g^*(w_{i-1})$+ 2; 2≤ $i \leq \frac{n+1}{2} - 4$ (if the case exists). Allocate the unutilized labels to unlabeled nodes namely $w_{\frac{n+1}{2}}, w_{\frac{n+1}{2}+1}, \dots, w_{n-2}$ simultaneously from {1,2,3,...n+2}.

In all the three cases discussed above it is visible that $\left|e_{g^*}(0) - e_{g^*}(1)\right| \leq 1$, which ensures that H^* is pcg.

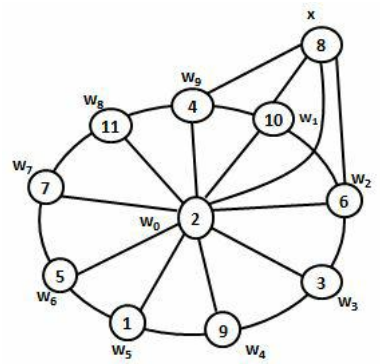

Figure 6. pcl of extension of rim vertex in W_9

Theorem 2.2.7. Extension of an arbitrary node of degree 2 of a flower graph Fl_n results in a pcg.

Proof. Let {v, v_i, u_i : 1≤ i ≤ n} be the node set of Fl_n where nodes v_i's and u_i's have degree 4 and 2 respectively. Let H^* be a graph produced by taking extension of an arbitrary node of degree 2. Without loss of generality, let us take the extension of node u_1 and let w be the freshly inserted node. Clearly the cardinalities of node set and edge set of H^* are respectively equal to 2n+2 and 4n+3. Consider a map g^*: V(H^*)→{ 1,2,...,2n+2} as given. Let g^* (v) = 2, $g^*(v_1)$ = 6, g^* (u_1) = 3 and g^* (w) = 1. Allocate available unutilized even labels simultaneously to v_i; 2 ≤ i ≤ n. Next, set g^* (u_i) = g^* (v_i) +1; 2 ≤ i ≤ n-1 and allocate the unutilized label to u_n. Following this pattern it is easy to see that H^* is pcg.

Remark 2.2.8. The graph formed by performing the extension of an arbitrary node of degree 4 in flower graph Fl_n allows a pcl. Here the count of edges will be 4n+ 5. Fix g^*(w) = 3, $g^*(v_1)$ = 6, $g^*(u_1)$ = 1 and $g^*(v_2)$ = 12, rest follow the above pattern.

Theorem 2.2.9. Extension of all vertices of path graph P_n allows a pcl.

Proof. Let u_1,u_2,…,u_n be the nodes of P_n. Let H^* be the graph acquired by taking extension of all nodes of P_n having node set V(H^*) = V(P_n) ∪{v_i : 1≤ i ≤ n} and edge set E(H^*) = E(P_n) ∪{u_iv_i: 1 ≤ i ≤ n} ∪ {v_iu_{i-1} : 2 ≤ i ≤ n}∪ {v_iu_{i+1}: 1≤ i ≤ n-1}. Clearly cardinalities of node set and edge set of H^* are equal to 2n and 4n-3 respectively. In order to define g^*:V(H^*) → { 1,2,...,2n}, we refer to theorem 2.2.3.

Theorem 2.2.10. Extension of all nodes of cycle graph C_n , n > 8 allows a pcl.

Proof: Let the node set of C_n be {u_1,u_2, …, u_n}. Let H^* be the graph produced by taking extension of all nodes of C_n having node set V(H^*) = V(C_n) ∪ {v_i : 1≤ i ≤ n } and edge set E(H^*) = E(C_n) ∪ {u_iv_i: 1≤ i ≤ n } ∪{$u_{i-1}v_i$: 2 ≤ i ≤ n }∪{ $u_{i+1}v_i$: 1≤ i ≤ n-1 }∪{ u_nv_1, u_1v_n}.Clearly the cardinalities of node set and edge set of H^* are equal to 2n and 4n. In order to define g^* : V(H^*) → { 1,2,...,2n}, we refer to theorem 2.2.3, except for the following few changes. Replace $g^*\left(u_{\frac{n}{2}+1}\right)$ = 9, $g^*\left(v_{\frac{n}{2}+1}\right)$ = 3, $g^*\left(u_{\frac{n}{2}+2}\right)$ =11, $g^*\left(v_{\frac{n}{2}+2}\right)$ = 5, $g^*\left(u_{\frac{n}{2}+3}\right)$ = 17, $g^*\left(v_{\frac{n}{2}+3}\right)$ = 15. (Similar pattern when n is odd). One can easily see that H^* is pcg.

*Remark 2.2.11***.** It is easy to deduce the pcl of the graph acquired by performing the extension of all the rim vertices of wheel graph W_n on similar lines with the above theorem.

Theorem 2.2.12. Extension of all vertices in a star graph allows a pcl.

Proof. Let {k_0, k_1,k_2, …,k_n} be the node set of star graph $K_{1,n}$ where k_0 represents the apex node and k_1,k_2, …,k_n represents the pendant nodes. Let H^* be the graph produced by performing extension of each node of $K_{1,n}$ and let u_0,u_1, u_2, …,u_n be the freshly inserted nodes. The node set and edge set of $H^*$are given by V(H^*) = V($K_{1,n}$) ∪ { u_0,u_1, u_2, …,u_n } and E(H^*) = E($K_{1,n}$) ∪ {k_iu_i : 1≤ i ≤ n} ∪ {k_0u_0 }∪ {k_0u_i : 1≤ i ≤ n}∪ {u_0k_i: 1≤ i ≤ n} respectively. Clearly the cardinalities of node set and edge set of H^* are equal to 2n+2 and 4n+1 respectively. Consider the map g^*: V(H^*) →{ 1,2,...,2n+2} defined as here. Fix $g^*(k_0)$ = 2, $g^*(k_1)$ = 3, $g^*(k_2)$ = 4, $g^*(k_3)$ = 8, $g^*(k_i)$ = $g^*(k_{i-1})$ +2; 4≤ i ≤ n, $g^*(u_0)$ = 6, g^* (u_1) = 1, g^* (u_2) = 5, $g^*(u_i)$ = $g^*(\,k_i) - 1$; 3 ≤ i ≤ n.

Note that GCD ($g^*(k_0)$, $g^*(k_i)$) >1; 2 ≤ i ≤ n,

GCD (g^* (k_0), $g^*(u_0)$) >1 and

GCD ($g^*(u_0)$, $g^*(k_i)$) > 1; 1≤ i ≤ n.

The edges due to above observation shall be labeled 0 and the remaining edges shall be labeled 1. We note here that $e_{g^*}(0)$ = 2n and $e_{g^*}(1)$= 2n+1. One can easily see that H^* is pcg.

Theorem 2.2.13. Extension of all nodes of bistar graph allows a pcl.

Proof. Let $B_{n,n}$ be the bistar having node set V($B_{n,n}$) = {u_0, v_0, u , v_i : 1≤ i ≤ n} where u_0, v_0 represents the apex nodes. Let H^* be the graph produced by performing extension of each node of $B_{n,n}$ and let v_0', u_0', v_i', u_i' ; 1≤ i ≤ n be the freshly added nodes.

The node set and edge set of H^* are given by V(H^*) = V($B_{n,n}$) ∪ {v_0', u_0', v_i', u_i' :1 ≤ i ≤ n} and E(H^*) = E($B_{n,n}$) ∪ {u_iu_i': 1≤ i ≤ n}∪ {v_iv_i': 1≤ i ≤ n}∪ {u_0v_0', v_0u_0', u_0u_0', v_0v_0'}∪ {u_0u_i', v_0v_i' : 1≤ i ≤ n} ∪ {$u_0'u_i$:1≤ i ≤ n} ∪ { $v_0'v_i$: 1 ≤ i ≤ n} respectively. The cardinality of node set of H^* is equal to 4n+ 4 and that of edge set is equal to 8n+ 5.

Consider the map g^*: V(H^*) → {1,2,...,4n+4} defined as here. $g^*(u_0)$ = 2, $g^*(v_0)$ = 4, $g^*(u_0')$ = 6, g^* (v_0') =1, $g^*(u_1)$ = 3, $g^*(u_1')$ = 12. Allot the unutilized even labels to u_i and u_i' ; 2≤ i ≤ n in any order. Next, $g^*(v_1)$ = 5, $g^*(v_i)$ = $g^*(v_{i-1})$ + 4; 2 ≤ i ≤ n and $g^*(v_i')$ = $g^*(v_i)$ +2; 1≤ i ≤ n.

Observe here that GCD (g^* (v_0'), g^* (v_i)) = 1; 1 ≤ i ≤ n,

GCD ($g^*(v_0)$, $g^*(v_i)$) = 1; 1 ≤ i ≤ n,

GCD ($g^*(v_0)$, $g^*(v_i')$) = 1; 1 ≤ i ≤ n,

GCD ($g^*(u_0)$, $g^*(v_o')$) =1,

GCD ($g^*(v_0)$, $g^*(v_0')$) = 1 and

GCD ($g^*(u_0)$, $g^*(u_1)$) = 1.

The edges due to above observation shall be labeled 1 and the remaining edges shall be labeled 0. Note here that $e_{g^*}(1)$ = 4n + 3 and $e_{g^*}(0)$ = 4n+ 2, which proves that H^* is pcg.

Vaidya et.al in [10] proved that one point union of C_3 allows a pcl. Motivated by this, we attempt to prove that one point union of K_4 also permits pcl.

Theorem 2.2.14. The point union of n-copies of K_4 allows a pcl.

Proof. Let H^* be the graph produced by taking the point union of n-copies of K_4 having node set V(H^*) = $\{k_0\}$ ∪ { k_{ij} : 1≤ i ≤ n, 1 ≤ j ≤ 3 } and edge set E(H^*) = { $k_0 k_{ij}$: 1 ≤ i ≤ n, 1 ≤ j ≤ 3} ∪ {$k_{i1}k_{i2}$,$k_{i1}k_{i3}$, $k_{i2}k_{i3}$:1 ≤ i ≤ n}. Clearly the cardinalities of node set and edge set of H^* are equal to 3n+1 and 6n respectively. Consider a map g^*: V(H^*) →{1,2,3,...,3n+1} as given. Choose the largest prime p such that 3p ≤ 3n+1. Fix $g^*(k_0)$ = 2p. Beginning with k_{11}, allocate all even labels simultaneously to the nodes k_{12}, k_{13}, k_{21},k_{22}, k_{23},.... in any fashion. Now we have two cases.

Case (i). When n is odd.

Allocate odd labels to unmarked nodes simultaneously from {1, 2, ..., 3n+1}.

Case (ii). When n is even.

Fix g^* ($K_{\frac{n}{2}3}$)=1, g^* ($K_{(\frac{n}{2}+1)\,1}$) =3, g^* ($K_{(\frac{n}{2}+1)2}$)= 9 and assign unutilized labels to unmarked nodes namely, $K_{(\frac{n}{2}+1)3}$, $K_{(\frac{n}{2}+2)1}$, $K_{(\frac{n}{2}+2)2}$, ..., k_{n3} simultaneously from unutilized labels out of {1,2,...,3n+1}.

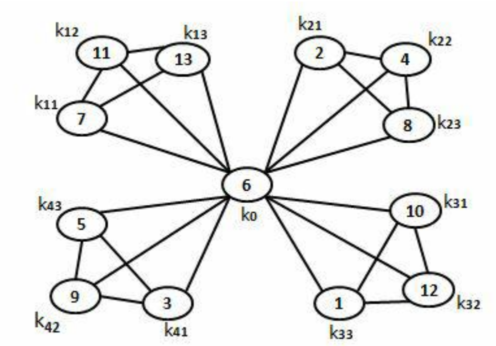

Figure 7. pcl of point union of 4 copies of K_4

2.3 Conjectures and Open Problems

We propose the following conjectures.

Conjecture 2.3.1. $P_n \odot \underline{K_m}$ permits a pcl.

Conjecture 2.3.2. If G is a k-regular pcg, then G$\odot\underline{K_1}$ is also a pcg.

Conjecture 2.3.3. Extension of an apex vertex of wheel graph does not permit a pcl.

Conjecture 2.3.4. The graph produced by performing the extension of each node of G allows a pcl, where G is a k-regular pcg.

In addition to the above conjectures, we also propose the following open problems.

Problem 2.3.5. Investigate whether the following graphs permit a pcl?

$C_n \odot \underline{K_m}$, $W_n \odot \underline{K_m}$, $Fl_n \odot \underline{K_m}$, $G_n \odot \underline{K_m}$

Problem 2.3.6. If G is a k-regular pcg, then does G $\odot$ $\underline{K_n}$ also pcg?

Problem 2.3.7. Investigate whether extension of each node of helm graph, flower graph, gear graph allows a pcl?

Problem 2.3.8. If G is a pcg then does graph produced by performing extension of each node of G also allow a pcl?

3. Conclusion

In the first subsection of the main results, we have accomplished that path, cycle, wheel, gear, and flower graphs are invariant under graph operation namely corona product, with $\underline{K_1}$, for pcl. Further we have also established that corona product of path P_n with $\underline{K_n}$ permits a pcl. In the second subsection we have established that graphs produced by performing extension of every node in path, cycle, wheel, star and bistar graph permit a pcl besides formulating some interesting conjectures and open problems.

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