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Analytical Solution of Some Nonlinear Equations Arising in Heat Transfer

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Abstract

In this paper modified variational iteration method is used to obtain analytic solution of some nonlinear equation arising in heat transfer. This method is developed by change in the formulation of variational iteration method. The solution is considered as an infinite series converging towards the exact solution rapidly.All computation work has been performed in MATHEMATICA..

Keywords: Modified variational iteration method, heat transfer, nonlinear equations

1. Introduction

Nonlinear equations arise in almost every engineering field e.g. Mechanics, fluid flows, civil engineering etc. It becomes challenging to find their exact solution most of the times.Heat transfer is another very important field where many applications are governed by nonlinear equations.Most of the nonlinear equations are difficult to solve analytically and we have to finally depend on an approximate solution.

Previously there are many methods applied to find analytical or numerical solution of nonlinear problems e.g. Homotopy perturbation method, Homotopy analysis method,Adomian decomposition method, variational iteration method, differential transform method and others.The main problem with these methods is that after few iterations the calculations become very large and computationally tedious. In this paper we have applied modifiedvaritional iteration method (MVIM)[1] to solve some nonlinear problems in heat transfer. This method is faster and more efficient than the original variational iteration method (VIM)[2]. It even converges more rapidly than VIM. It is developed by changing the formulation of VIM.We have obtained the exact solution of the considered nonlinear problem using MVIM.

This paper is organized as follows: Section 2 gives a brief introduction about modified variational method. In Section 3, two nonlinear problems arising in heat transfer have been solved by modified variationalmethod and lastly conclusions are drawn in Section 4.

2. Modified Variational Iteration Method:

In this section we will give a brief description of modified variationalmethod[1]. Let us consider a nonlinear problem

$$Lu(x,t)+Ru(x,t)+Nu(x,t)=g(x,t) \tag{1}$$

$$u(x,0)=f_0(x)$$

$$\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0}=f_1(x)$$

$$\left.\frac{\partial^{s-1}u(x,t)}{\partial t^{s-1}}\right|_{t=0}=f_{s-1}(x)$$

where $L = \frac{\partial^s}{\partial t^s}, s = 1,2,3...$ is the highest partial derivative with respect to t, $Ru(x,t)$ is a linear operator and $Nu(x,t)$ is the nonlinear term. $Ru(x,t)$ and $Nu(x,t)$ are free of partial derivatives with respect to t and $g(x,t)$ is the non-homogeneous term.

In the variational iteration method we use the following iteration method:

$$U_{n+1}(x,t) = U_n(x,t) + \int_0^\tau \lambda \left\{ LU_n + R\tilde{U}_n + N\tilde{U}_n \right\} d\tau \tag{2}$$

where λ is called Lagrange multiplier which is determined by variational theory and c is considered a restricted variation i.e. $\delta\tilde{U}_n = 0$.

Now taking variations with respect to $\tilde{U}_n$

$$\delta U_{n+1}(x,t) = \delta U_n(x,t) + \delta\int_0^\tau \lambda \left\{ LU_n + R\tilde{U}_n + N\tilde{U}_n \right\} d\tau \tag{3}$$

$$\delta U_{n+1}(x,t) = \delta U_n(x,t) + \delta\int_0^\tau \lambda LU_n d\tau \tag{4}$$

which gives

$$\lambda = -\frac{(t-\tau)^{(s-1)}}{(s-1)!} \tag{5}$$

Now substituting the value of λ from equation (5) to (2) we get

$$U_{n+1}(x,t) = U_n(x,t) + \int_0^\tau -\frac{(t-\tau)^{(s-1)}}{(s-1)!} \left\{ LU_n + R\tilde{U}_n + N\tilde{U}_n \right\} d\tau \tag{6}$$

Equation (6) becomes the formulation of variational iteration method(VIM). In the modified version of variational iteration method (MVIM)[1] we use the following iteration formula

$$U_{n+1}(x,t) = U_n(x,t) + \int_0^t \lambda \left\{ R(U_n - U_{n-1}) + (G_n - G_{n-1}) - \left(g_{ns}(x)\tau^{ns} + g_{ns+1}(x)\tau^{ns+1} ... + g_{s\{(n+1)-1\}}(x)\tau^{s\{(n+1)-1\}} \right) \right\} d\tau \tag{7}$$

where λ has the same value as in equation (5) , $G_n(x,t)$ is a polynomial of degree $s(n+1)-1$ and is obtained from $NU_n(x,t) = G_n(x,t) + O\left(t^{s(n+1)}\right)$ and $g_n(x)$ is obtained by Taylor's series expansion of $g(x,t)$ where $g(x,t) = \sum_{n=0}^{\infty} g_n(x) t^n$.

There are many versions of modified vatiational iteration method in the literature but the formulation (7)is simpler, easier to apply , faster and more efficient than the others. In the next section we will find out the analytical solution of two nonlinear equations arising in heat transfer using modified variational iteration method (MVIM) described here.

3. Numerical Examples:

We are considering two applications of heat transfer which are governed by nonlinear equations.

3.1 Case 1:Cooling of a lumped system by combined convection and radiation : Let us take a system [3,4]with surface area$A$, density ρ, volume V, specific heat c , emissivity E and intial temperature T_0. At $t = 0$ the system is exposed to an environment of convective heat transfer with the coefficient of h and the temperature T_a. There is also a loss of the heat of the system due to radiation and the effective sink temperature is T_s. Cooling of such system is governed by the following nonlinear equation:

$$cV\frac{dT}{dt}+hA\left(T-T_a\right)+E\sigma A\left(T^4-T_s^4\right)=0 \qquad (8)$$
$$T(0)=T_0$$

Employing the following parameters:

$$\theta=\frac{T}{T_0}, \quad \theta_a=\frac{T_a}{T_0}, \quad \theta_s=\frac{T_s}{T_0}, \quad \tau=\frac{hAt}{\rho cV}, \quad \varepsilon=\frac{E\sigma T_0^3}{h}$$

The problem (8) is reduced to

$$\frac{d\theta}{d\tau}+\left(\theta-\theta_a\right)+\varepsilon\left(\theta^4-\theta_s^4\right)=0 \qquad (9)$$
$$\theta\left(0\right)=1$$

Let us consider the case where $\theta_a=\theta_s=0$. Then equation (9) becomes

$$\frac{d\theta}{d\tau}+\varepsilon\theta^4+\theta=0 \qquad (10)$$
$$\theta\left(0\right)=1$$

The exact solution of this problem is given by following relation

$$\tau=\frac{1}{3}Ln\frac{1+\varepsilon\theta^3}{\left(1+\varepsilon\right)\theta^3} \qquad (11)$$

By expanding around $\tau=0$ equation (11) gives the following series form of solution :

$$\theta\left(\tau\right)=1+\left(-1-\varepsilon\right)\tau+\left[\frac{1}{2}+\frac{5}{2}\varepsilon+2\varepsilon^2\right]\tau^2+\left[-\frac{1}{6}-\frac{7}{2}\varepsilon-8\varepsilon^2-\frac{14}{3}\varepsilon^3\right]\tau^3+... \qquad (12)$$

Now we have applied modified variational iteration method (MVIM)[1] to (10). Here $s=1, R\theta(\tau)=\theta(\tau), N\theta(\tau)=\varepsilon[\theta(\tau)]^4$ which leads to $\lambda=-1$ and $g(\tau)=0$.Hence the iteration formula (7) for applying MVIM to this problem becomes:

$$\theta_{n+1}=\theta_n-\int_0^\tau\left\{\left(\theta_n\left(k\right)-\theta_{n-1}\left(k\right)\right)+\left(G_n\left(k\right)-G_{n-1}\left(k\right)\right)\right\}dk \qquad (13)$$

where$\theta_{-1}=0, \theta_0=1$ and G_n can be determined from the following equation:

$$\varepsilon[\theta(\tau)]^4=G_n(\tau)+O\left(\tau^{n+1}\right)$$

By applying formula (13) we get the following result:

$$\theta_1=1+\tau(-1-\epsilon)$$
$$\theta_2=1+\tau(-1-\epsilon)+\frac{1}{2}\tau^2(1+5\epsilon+4\epsilon^2)$$
$$\theta_3=1+\tau(-1-\epsilon)+\frac{1}{2}\tau^2(1+5\epsilon+4\epsilon^2)+\frac{1}{6}\tau^3(-1-21\epsilon-48\epsilon^2-28\epsilon^3)$$
$$\theta_4=1+\tau(-1-\epsilon)+\frac{1}{2}\tau^2(1+5\epsilon+4\epsilon^2)+\frac{1}{6}\tau^3(-1-21\epsilon-48\epsilon^2-28\epsilon^3)+\frac{1}{24}\tau^4(1+85\epsilon+420\epsilon^2+616\epsilon^3+280\epsilon^4)$$

These are the approximations at four iterations. We can find more approximations by evaluating more iterations. Now if we compare our approximation at last iteration θ_4 to the series form of the solution θ in equation (12) it is matching with the first four terms of the series exactly. By looking at first four approximations by MVIM we can see that at every iteration we are getting one more term of series (12). Hence we can safely say that if we continue to evaluate more iterations same series will be formed as (12).

Previously many other methods have been applied on the same problem such as homotopy perturbations method and variational iteration method in [5], differential transform method in [3], Double trials method in [6] etc. Clearly MVIM gets better result than other methods as we are getting the same series of the solution as by the exact solution whereas in other methods we get an approximate solution.

3.2Case 2: Transient heat conduction in a slab with temperature dependent thermal conductivity: Consider a slab with unit length i.e. $0\le x\le 1$ and temperature dependent thermal conductivity $k(T)=1+T^2$. The heat conduction in this slab is governed by the following system[7]:

$$\frac{\partial}{\partial x}\left(k(T)\frac{\partial T}{\partial x}\right)+f\left(x,t\right)=\frac{\partial T}{\partial t} \quad , \ 0\le x\le 1,\ t>0$$

with conditions

$$T\left(0,t\right)=0\ ,T\left(1,t\right)+\frac{\partial T\left(1,t\right)}{\partial x}=2e^{t}$$

$$T\left(x,0\right)=x$$

and

$$k\left(T\right)=1+T^{2}\ ,\ f\left(x,t\right)=xe^{t}\left(1-2e^{2t}\right)$$

The exact solution of the problem is $T\left(x,t\right)=xe^{t}$. Now, for this problem we have $s=1, R\{T(t)\}=0, N\{T(t)\}=-\frac{\partial}{\partial x}\left(k(T)\frac{\partial T}{\partial x}\right)$ which gives $\lambda=-1$ and $g(t)=\mathrm{x}\mathrm{e}^{\mathrm{t}}(1-2e^{2t})$.Hence the MVIM iteration formula for this problem becomes:

$$T_{n+1}=T_{n}-\int_{0}^{t}\left\{\left(T_{n}\left(k\right)-T_{n-1}\left(k\right)\right)+\left(G_{n}\left(k\right)-G_{n-1}\left(k\right)\right)\right\}dk \qquad (14)$$

where$T_{-1}=0, T_{0}=1$ and G_{n} can be determined from the following equation:

$$-\frac{\partial}{\partial x}\left(k(T)\frac{\partial T}{\partial t}\right)=G_{n}(t)+O\left(t^{n+1}\right)$$

andg_{n} is obtained by the Taylor's series expansion of $g(t)=\mathrm{x}\mathrm{e}^{\mathrm{t}}(1-2e^{2t})$ around $t=0$

$$\sum_{n=0}^{\infty}g_{n}t^{n}=xe^{t}\left(1-2e^{2t}\right)$$

By applying formula (14) we get the following result:

$$T_{1}=x+tx$$

$$T_{2}=x+tx+\frac{t^{2}x}{2}$$

$$T_{3}=x+tx+\frac{t^{2}x}{2}+\frac{t^{3}x}{6}$$

$$T_{4}=x+tx+\frac{t^{2}x}{2}+\frac{t^{3}x}{6}+\frac{t^{4}x}{24}$$

By looking at the output of iterations above we can assume that if we take limit $n\rightarrow\infty$ the series obtained will be the expansion of function xe^{t} which is the exact solution of the problem. This problem is solved by operational Tau method in [7] and only approximate solution has been obtained. Hence it clearly shows that modified variational method should be preferred for solution of such type of problems [8-13].

4. Conclusion:

In this paper we have achieved exact solutions of two nonlinear problems arising in heat transfer using modified variation iteration method (MVIM). In comparison to other methods that have been used in solving same type of problems, MVIM, if converges, gives exact solution most of the times. It is also less complicated, faster and requires less supplementary conditions

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