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Nonexpansive Mappings Andextremal Points in Hyperconvex Metric Spaces

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Abstract

Aronszajn and Panitchpakdi developed hyperconvex metric spaces to expand Hahn-theorem Banach's beyond the real line to more generic spaces. The aim of this short article is to collect and combine basic notions and results in the fixed point theory in the context of hyperconvex metric spaces. In this paper, we first introduce the definitions of hyperconvex metric spaces, nonexpansive retract, externally hyperconvex and bounded subsets, and admissible subsets. We shall review and explore some fundamental characteristics of hyperconvexity. Next, we introduce the Knaster–Kuratowski and Mazurkiewicz (KKM) theory in hyperconvex metric spaces and related results. Furthermore, we find the relationship between extremal points and hyperconvexity and related properties. Furthermore, we have highlighted some known consequences of our main results.Finally, we prove the characterization of the generalized metric KKM mapping principle in hyperconvex metric spaces. It is also aimed at showing that there are still enough rooms for several researchers in this interesting direction and a huge application potential. In the concluding part of the article, we have finally reiterated the well-demonstrated fact that the results presented in this article can easily be rewritten as a nonexpansive retract by making some straightforward simplifications, and it will be an inconsequential exercise, simply because the additional properties are obviously unnecessary.

Keywords: Hyperconvex metric space, externally hyperconvex,admissiblesubset,injective metric space,finiteintersection property.

Introduction

Aronszajn and Panitchpakdi introduced the concept of hyperconvexity by demonstrating that a hyperconvex space is an absolute retraction, i.e., it is a nonexpanding retraction of any metric space in which it is isometrically contained. The related linear concept is well established and is attributed to Goodner and Nachbin. The reader may refer to Lacey [8] for further information on that linear theory. The nonlinear theory is still in its infancy. Isbell created a natural hyperconvex hull for every metric space. Recent interest in such spaces stems from the independent proofs by Sine [1] and Soardi [12] that the fixed point condition for nonexpansive mappings holds in bounded hyperconvex spaces. Numerous intriguing findings [6, 7, 9, 10] have been shown to hold in hyperconvex spaces. We are aware of the significance of the well-known Fan-KKM principle in the study of nonlinear analysis, particularly in the study of topology fixed point theory. Sine and Soardi separately demonstrated the importance of the connection between hyperconvex metric spaces and nonexpansive mappings [1,13].

Additionally, bear in mind that Jawhari et al. demonstrated that Sine and Soardi's fixed point theorem is identical to the traditional Tarski fixed point theorem in fully ordered sets. This is accomplished via the concept of generalised metric spaces. As a result, hyperconvexity should be understood and appreciated in a more abstract sense. We shall review and explore some fundamental characteristics of hyperconvexity in this study. Additionally, we will examine KnasterMazurkiewicz mappings in short KKM-maps and establish an equivalent of Ky Fan's fixed point theorem, which may be seen as an extension of Brouwer and Schauder's fixed point theorems. To our knowledge, this is the first time that such theorems have been attempted to be proved in a metric space context.

The connection between hyperconvex metric spaces and nonexpansive maps is critical, as Sine [1] and Soardi [2] demonstrate separately. But on the other hand, we are well aware of the significance of the well-known Fan-KKM principle in the research of nonlinear analysis, particularly in the study of topological fixed point theory, as shown in [3, 4, 14, 15] and references thereto. Khamsi presented a hyperconvex variant of Fan's best approximation theorem with single-valued maps and the Schauder–Tychonoff fixed point theory in [16]. The entire point of such an article is in that direction, namely, to provide a thorough article on KKM theory in hyperconvex spaces and its frameworks for fixed point theorems, to classify the KKM principle in hyperconvex spaces, to receive Fan's minimax principle in hyperconvex spaces, to establish the existence of saddle points, to establish the intersection of sets, and to establish the existence of Nash equilibria. To explore the KKM theory for hyperconvex spaces, it is necessary to first review certain notations and fundamental facts regarding hyperconvex spaces that will be utilized later in the article.

2 Notation and basic definitions

For convenience, metric spaces shall be represented as(M,d), or simply M, where M denotes the space and d is the distance on *M.* For us, the primary elements in a metric space will be closed balls indicated by $B(x,r)$, which stands for the closed ball with center x and radius $r \geq 0$. Additionally, the following notation is common when working with metric spaces and will be utilized throughout this article. Assume that M is a metric space, and that $x \in M$ and A and B are subsets of M;then

$$\begin{aligned} r_x(A) &= \mathrm{s} \quad \{d(x,y): y \in A\} \\ r(A) &= \mathrm{i} \quad \{r_x(A): x \in M\} \\ R(A) &= \mathrm{i} \quad \{r_x(A): x \in A\} \\ \mathrm{diam}(A) &= \mathrm{s} \quad \{d(x,y): x, y \in A\} \\ \mathrm{dist}(x, A) &= \mathrm{i} \quad \{d(x,y): y \in A\}, \\ \mathrm{dist}(A, B) &= \mathrm{i} \quad \{d(x,y): x \in A, y \in B\}, \\ C(A) &= \{x \in M: r_x(A) = r(A)\} \\ C_A(A) &= \{x \in A: r_x(A) = R(A)\} \end{aligned}$$

$\mathrm{cov}(A) = \bigcap\{B: B$ is a closed ball containing $A\}$ where $r(A)$ is the radius of A relative to M, $\mathrm{diam}(A)$ is the diameter of A, $R(A)$ is the Chebyshev radius of A, and $\mathrm{cov}(A)$ is the admissible cover of A.

Aronszajn and Panitchpakdi developed hyperconvex metric spaces to expand Hahn-theorem Banach's beyond the real line to more generic spaces, see [11,18-19]. As a consequence, they established metric criteria ensuring this extension and designated spaces fulfilling theserequirements hyperconvex metric spaces. The basic definition is given by Aronszajn and Panitchpakdi[17]. We shall begin by defining hyperconvexity.

Definition 2.1 A metric space M is said to be hyperconvex if given any family $\{x_\alpha\}$ of points of M and any family $\{r_\alpha\}$ of real numbers satisfying

$$d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$$

then

$$\bigcap_\alpha B(x_\alpha, r_\alpha) \neq \emptyset$$

where $B(x, r)$ denotes the closed ball centered at $x \in X$ with radius $r \geq 0$.

The recent interest into hyperconvexity goes back to the results of Sine andSoardi.

Definition 2.2 If H is a bounded hyperconvex metric space and $T: H \to H$ is nonexpansive, i.e., $d(T(x), T(y)) \leq d(x, y)$ for any $x, y \in H$, then there exists a fixed point $x \in H$, i.e., $T(x) = x$. Moreover, the fixed point set Fix (T) is hyperconvex and, consequently, is a nonexpansive retract of H.

Definition 2.3 A subset A of a metric space X is called externally hyperconvex (cf. [1]) if for any collection of balls $\{B(x_i, r_i)\}_{i \in I}$ in X with $d(x_i, x_j) \leq r_i + r_j$ and $d(x_i, A) \leq r_i$ we have $A \cap \bigcap_i B(x_i, r_i) \neq \emptyset$.

A subset of a metric space that has this characteristic is often referred to as a hyperconvex set. Hyperconvexity gets its name from the fact that when the ball intersecting condition remains true for every pair of balls, the metric space was seen to be (metrically) convex. They are also referred to as injective metric spaces in the literature, while hyperconvexBanach spaces are generally referred to as $\mathcal{P}_1$ spaces [5,20]. The research of normed spaces that fulfill this characteristic and others was critical in the mid-twentieth century, and the interested reader may see [20]. For additional information on hyperconvex metric spaces' injective character, see [18,11] or [19, Section 4].

The following statement establishes a basic truth about hyperconvex spaces, which is shown in [18,19],see also [11, Proposition 4.4].

Proposition 2.4 If M is a hyperconvex metric space then it is complete. Following that, we outline some metric characteristics of hyperconvex spaces that have a significant impact on the structure of hyperconvex spaces and are widely used in metric fixed point theory; for proofs, see [19, Lemma 3.3] or [11, Lemma 4.1].

Lemma 2.5 Let A be a bounded subset of a hyperconvex metric space M.Then

1 $\operatorname{cov}(A) = \cap \{B(x, r_x(A)) : x \in M\}$.

2 $r_x(\operatorname{cov}(A)) = r_x(A)$, for any $x \in M$.

3 $r(\operatorname{cov}(A)) = r(A)$.

4 $r(A) = \frac{1}{2}\operatorname{diam}(A)$.

5 $\operatorname{diam}(\operatorname{cov}(A)) = \operatorname{diam}(A)$

6 If $A = \operatorname{cov}(A)$then $r(A) = R(A)$.

Definition 2.6 Let M be a metric space. $A \subseteq M$is said to be an admissible subset of M if $A = \operatorname{cov}(A)$. The collection of all admissible subsets of M is then denoted by $\mathcal{A}(M)$.

The word hyperconvex has certain pejorative consequences.

Proposition 2.7 Let (X, d) be a hyperconvex space and $\{A_i\}_{i\in I}$ a family of pairwise intersecting externally hyperconvex subsets such that one of them is bounded. Then $\bigcap_{i\in I} A_i \neq \emptyset$

Proposition 2.8 (Theorem 4 in [17]). A metric space (X, d) is injective if and only if it is hyperconvex.

Proposition 2.9A Banach space is said to be hyperconvex if and only if it is linearly isometric to C(K), where C(K) denotes the space of all continuous real functions defined on any stonian space K.

Hyperconvex spaces may exhibit certain peculiar properties.A hyperconvex subset does not have to be convex.Additionally, convex sets in linear spaces may be non hyperconvex.For hyperconvex sets, which are intersections of balls, more compelling parallels exist.

3. The KKM theory in hyperconvex metric spaces

The fundamental result of asserts that a metric space M is hyperconvex if and only if it is injective. Thus M is hyperconvex if given any two metric spaces X and Y with Y a subspace of X, and any nonexpansive mapping $f: Y \to M$, then f has a nonexpansive extension $\tilde{f}: X \to M$. An admissible subset of M is a set of the form

$$\bigcap_i B(x_i; r_i)$$

where$\{B(x_i; r_i)\}$ is a family of closed balls centered at points $x_i \in M$ with respective radii r_i. It is quite easy to see that an admissible subset of a hyperconvex metric space is hyperconvex. In what follows we use $\mathcal{A}(M)$ to denote the family of all nonempty admissible subsets of M.

While the intersection of two admissible subsets of a hyperconvex space is again admissible, the intersection of two hyperconvex subspaces of a hyperconvex space is not always hyperconvex, even if one of them is admissible. The following, however, is accurate.

Lemma 3.1 Let H be a hyperconvex metric space. Suppose $E \subset H$ is externally hyperconvex relative to H and suppose A is an admissible subset of H. Then $E \cap A$ is externally hyperconvex relative to H.

Proof. Suppose $\{x_\alpha\}$ and $\{r_\alpha\}$ satisfy $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$ and

$\operatorname{dist}(x_\alpha, E \cap A) \leq r_\alpha$. Since A is admissible, $A = \cap_{i \in I} B(x_i; r_i)$ and since $\operatorname{dist}(x_\alpha, E \cap A) \neq \emptyset$ it follows that $d(x_\alpha, x_i) \leq r_\alpha + r_i$ for each $i \in I$. Also, since $A \subset B(x_i; r_i)$, it follows that $\operatorname{dist}(x_i, E \cap A) \leq r_i$ and that $d(x_i, x_j) \leq r_i + r_j$ for each $i, j \in I$. Therefore by external hyperconvexity of E

$\left(\cap_i B(x_i; r_i)\right)\left(\cap_\alpha B(x_\alpha, r_\alpha)\right) \cap E = \cap_\alpha B(x_\alpha, r_\alpha) \cap (A \cap E) \neq \emptyset.$

This leads to the following.

Theorem 3.2 Let $\{H_i\}$ be a descending chain of nonempty externally hyperconvex subsets of a bounded hyperconvex space H.Then$\cap_i H_i$ is nonempty and externally hyperconvex in H.

Proof. Assures that $D := \cap_i H_i \neq \emptyset$. To see that D is externally hyperconvex. let $\{x_\alpha\} \subset H$ and $\{r_\alpha\} \subset \mathbb{R}$ satisfy $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$ and
$\operatorname{dist}(x_\alpha, D) \leq r_\alpha$. Since H is hyperconvex we know thal $A := \cap_\alpha B(x_\alpha; r_\alpha) \neq \emptyset$
Also, since $\operatorname{dist}(x_\alpha, D) \leq r_\alpha$ we have $\operatorname{dist}(x_\alpha, H_i) \leq r_\alpha$ for each i, so by external

hyperconvexity of H_i we conclude $A \cap H_i \neq \emptyset$ for each i. By Lemma 3.1$\{A \cap H_i\}$ is descending chain of nonempty hyperconvex subsets of H, so we have $\cap_i (A \cap H_i) = A \cap D \neq \emptyset$.

The following KKM-theorem is due to Khamsi [16, Theorem 4]:

Theorem 3.3. Let H be a hyperconvex space and $X \subset H$ a subset. Let $G: X \multimap H$ be a KKM map such that $G(x)$ is closed for any $x \in X$ and $G(x_0)$ is compact for some $x_0 \in X$. Then we have

$$\bigcap_{x \in X} G(x) \neq \emptyset$$

for a map $G: X \multimap Y$, we denote $x \in G^{-1}(y)$ iff $y \in G(x)$ where $x \in X$ and $y \in Y$. Let $\mathbb{C}(X, Y)$ denote the class of single-valued continuous maps $f: X \to Y$.

In this section, we deduce useful generalized forms of the KKM type theorems. From Theorem 3.3, we have the following:

Theorem 3.4 Let H be a hyperconvex space, $X \subset H$, and $G: X \multimap H$ a KKM map with compactly closed values. Then for every compact hyperconvex subsets $K_0, H_0 \subset H$, we have

$$(K_0 \cap H_0) \cap \bigcap_{x \in (K_0 \cap H_0) \cap X} \{G(x): x \in x \in (K_0 \cap H_0) \cap X\} \neq \emptyset \qquad (1)$$

Proof. Define $G_0(x) = G(x) \cap (K_0 \cap H_0)$for $x \in (K_0 \cap H_0) \cap X$. Then
$G_0: (K_0 \cap H_0) \cap X \multimap H_0$ is well-defined.

If $K_0 \cap H_0 = \emptyset$. Then by Theorem 3.3, (1) is true trivially.

Therefore, without restriction of the generality, we can suppose that $K_0 \cap H_0 \neq \emptyset$.

Consider $(K_0 \cap H_0, K_0 \cap H_0 \cap X, G_0)$ instead of (H, X, G) in Theorem 3.3. Then all of the requirements are satisfied. Therefore, we have

$$\bigcap_{x \in (K_0 \cap H_0) \cap X} \{G_0(x): x \in (K_0 \cap H_0 \cap X\} = \bigcap_{x \in (K_0 \cap H_0) \cap X} \{G(x) \cap (K_0 \cap H_0) : x \in (K_0 \cap H_0 \cap X\}$$

$$= \quad (K_0 \cap H_0) \cap \bigcap_{x \in (K_0 \cap H_0) \cap X} \{G(x): x \in x \in (K_0 \cap H_0) \cap X\} \neq \emptyset$$

This completes our proof.

From Theorem 3.4, we have the following:

Theorem 3.5. Let H be a hyperconvex space, $X \subset H, Y$ a topological space,
$t \in \mathbb{C}(H, Y), G: X \multimap Y$ a map,andK_1 , K_2 be two nonempty compact subsets of Y. Suppose that
(3.1) for each $x \in X, G(x)$ is compactly closed;
(3.2) $t^{-1}G: X \multimap H$ is a KKM map; and

(3.3) for any $N \in \langle X\rangle$, there exists a compact hyperconvex subset $L_N \subset H$ containing N such that $t(L_N) \cap \bigcap\{G(x): x \in L_N \cap X\} \subset K_1$ and $t(L_N) \cap \bigcap\{G(x): x \in L_N \cap X\} \subset K_2$
Then we have

$$\overline{t(H)} \cap K_1 \cap K_2 \cap \bigcap \{G(x): x \in X\} \neq \emptyset$$

Proof. Since $t(L_N) \cap \bigcap\{G(x): x \in L_N \cap X\} \subset K_1$ and $t(L_N) \cap \bigcap\{G(x): x \in L_N \cap X\} \subset K_2$ which implies that $(L_N) \cap \bigcap\{G(x): x \in L_N \cap X\} \subset K_1 \cap K_2$.

Suppose that the conclusion does not hold. Since $\overline{t(H)} \cap K_1 \cap K_2$ is compact and contained in $\bigcup\{Y \setminus G(x): x \in X\}$, by (3.1), there exists an $N \in \langle X\rangle$ such that

$$\overline{t(H)} \cap K_1 \cap K_2 \subset \bigcup_{x \in N} (Y \setminus G(x))$$

For the $L_N \subset H$in (3.3), this implies

$$L_N \cap \bigcap_{x \in L_N \cap X} t^{-1}G(x) \cap t^{-1}(K_1 \cap K_2) = \emptyset$$

However, by (3.3), we have

$$L_N \cap \bigcap_{x \in L_N \cap X} t^{-1}G(x) \subset t^{-1}(K_1 \cap K_2)$$

Therefore, we have

$$L_N \cap \bigcap_{x \in L_N \cap X} t^{-1}G(x) = \emptyset$$

which contradicts Theorem 3.4. This completes our proof.
REMARK. For $H = Y, t = 1_H$ the identity map of H, and $K_1 \cap K_2 = G(x_0)$ for some $x_0 \in X$, Theorem 3.5 reduces to Theorem 3.3 . Therefore, in Theorem 3.3, G may have compactly closed values.Borkowski M, Bugajewski D, Przybycień H.

4. Extremal PointsAndHyperconvexity

Borkowski M, Bugajewski D, Przybycie H, investigate linear hyperconvex spaces with extremal points of their unit balls [21]. They prove that only in the case of a plane (and obviously a line) is there a strict connection between the number of extremal points of the unit ball and the hyperconvexity of space. At the beginning of this section, we shall consider linear space. First, we prove the following:

Proposition4. 1. Let $\|\cdot\|$ be a norm in $\mathbb{R}^2$ such that the closed unit ball in this norm has exactly four extremal points. Then $\mathbb{R}^2$ with this norm is a hyperconvex metric space.

Proof: Let$z_i = (x_i, y_i)$, where $i = 1, \dots, 4$, denote the extremal points of the closed unit ball $\bar{B}$. We may assume that $z_1 = -z_3$ and $z_2 = -z_4$. Let us denote the maximum norm in $\mathbb{R}^2$ by $\|\cdot\|_m$.

Consider the linear mapping $T: (\mathbb{R}^2, \|\cdot\|_m) \to (\mathbb{R}^2, \|\cdot\|)$ given by the matrix

$$M(T) = \begin{bmatrix} \frac{1}{2}(x_1 + x_2) & \frac{1}{2}(x_1 - x_2) \\ \frac{1}{2}(y_1 + y_2) & \frac{1}{2}(y_1 - y_2) \end{bmatrix}$$

Put $u_1 = (1,1), u_2 = (1,-1), u_3 = (-1,1), u_4 = (-1,-1)$. Obviously, we have $Tu_i = z_i$ for $i = 1, \dots, 4$. Let us denote by $\bar{B}_m$ the closed unit ball in the space $(\mathbb{R}^2, \|\cdot\|_m)$. Then $\bar{B}_m = \operatorname{conv}\{u_i : i = 1, \dots, 4\}$ and $T(\bar{B}_m) = \bar{B}$.

Thus T is anonsingular linear mapping of norm 1. Further, there exists $T^{-1}$and $\|T^{-1}\| = 1$, so T is an isometry. Hence $(\mathbb{R}^2, \|\cdot\|_m)$ is hyperconvex and the proof is complete. ∎

The question is now as follows: does there exist a norm in $\mathbb{R}^2$ such that the closed unit ball in this normed space has more than 4 extremal points and this space is hyperconvex? The following result gives the answer.

Proposition 4.2. Let there be given a norm in $\mathbb{R}^2$ such that the closed unit ball in this space has more than four extremal points. Then this space is not hyperconvex.

As a corollary from Propositions 4.1 and 4.2 we obtain the following characterisation.

Theorem 4.3. A space $(\mathbb{R}^2, \|\cdot\|)$ is hyperconvex if and only if the closed unit ball in this space has exactly four extremal points.

The space $\mathbb{R}^3$ has quite different character. Namely, we shall prove the following

Proposition 4.4. For every even number $n \geqslant 6$ there exists a norm $\|\cdot\|_n$ in $\mathbb{R}^3$ such that:

a) the closed unit ball in $(\mathbb{R}^3, \|\cdot\|_n)$ has exactly n extremal points.
b) The space $(\mathbb{R}^3, \|\cdot\|_n)$ is not hyperconvex.

5. Generalized forms of the KKM type Theorems

Let X be a nonempty set. We denote by $\mathcal{F}(X)$ and 2^X the family of all nonempty finite subsets of X and the family of all subsets of X, respectively. If A is a subset of a linear space E, the notation ' $\operatorname{conv}(A)$ ' always means the convex hull of A.

Definition 5.1. Let X be any nonempty set and let M be a metric space. A set-valued mapping $G: X \to 2^M \setminus \{\emptyset\}$ is said to be a generalized metric KKM mapping (GMKKM) if for eachnonempty finite set $\{x_1, \dots, x_n\} \subset X$, there exists a set $\{y_1, \dots, y_n\}$ of points of M, not necessarily all different, such that for each subset $\{y_{i_1}, \dots, y_{i_k}\}$ of $\{y_1, \dots, y_n\}$ we have

$$\mathrm{co}\{y_{i_j} : j = 1, \dots, k\} \subset \bigcup_{j=1}^{k} G\left(x_{i_j}\right)$$

Definition 5.2. Let X be a nonempty subset of a metric space M.Suppose$G: X \to 2^M$ is a set-valued mapping with nonempty values. Then G is said to be a metric KKM (MKKM) mapping if for each finite subset $F \in \mathcal{F}(X)$, $\mathrm{co}(F) \subset \bigcup_{x \in F} G(x)$.

Now, we give a characterization of the generalized metric KKM mapping principle in hyperconvex metric spaces.

Theorem 5.3. Let X be a non-empty set and M be a hyperconvex metric space. Suppose $G: X \to 2^M \setminus \{\emptyset\}$ is a set-valued mapping with nonempty closed values and suppose there exists$x_0 \in X$ such that $G(x_0)$ is compact. Then $\bigcap_{x \in X} G(x) \neq \emptyset$ if and only if the mapping G is a generalized metric KKM mapping.

Proof. Necessity: Since $\bigcap_{x \in X} G(x) \neq \emptyset$, it follows that the family $\{G(x): x \in X\}$ has the finite intersection property. Since $G(x)$ is closed for each $x \in X$ it is finitely metrically closed.

First of allwe shall prove the following result:

The family $\{G(x) : x \in X\}$ has the finite intersection property if and only if the mapping G is a generalized metric KKM mapping.

If the family $\{G(x): x \in X\}$ has the finite intersection property then for each finite subset $\{x_1, \dots, x_n\} \subset X, \bigcap_{i=1}^{n} G(x_i) \neq \emptyset$.

Take any point $x^* \in \bigcap_{i=1}^{n} G(x_i)$ and set $y_i \equiv x^*$ for $i = 1, \dots, n$
Then for any $1 \leq k \leq n$ and any subsequence $y_{i_1}, \dots, y_{i_k}$, it follows that

$\mathrm{co}(\{y_{ij} : j = 1, \dots, k\}) = co(\{x^*\}) = \{x^*\} \subset \cup_{i=1}^{k} G\left(x_{i_j}\right)$.

This proves that G is a GMKKM mapping.
On the other hand we may suppose that $G: X \to 2^M \setminus \{\emptyset\}$ is a GMKKM mapping and suppose the family $\{G(x): x \in X\}$ does not have the finite intersection property. Then there exists a nonempty finite set $\{x_1, \dots, x_n\}$ for which $\bigcap_{i=1}^{n} G(x_i) = \emptyset$. Since G is a GMKKM mapping there exist corresponding points $y_1, \dots, y_n$ of M such that for each subsequence $y_{i_1}, \dots, y_{i_k}$, we have $\mathrm{co}(\{y_{i_1}, \dots, y_{i_k}\}) \subset \bigcup_{j=1}^{k} G\left(x_{i_j}\right)$. Since M is hyperconvex

there exists a nonexpansive retraction $r: M_\infty \to M$. In particular, if we identify M with its isometric copy in the Banach space M_∞ then r maps the linear span L of points $\{y_1, \dots, y_n\}$into M. Let $Y := \mathrm{co}(\{y_1, \dots, y_n\})$and $S := \mathrm{conv}(\{y_1, \dots, y_n\})$.

Then $r(S) \subset M$ (indeed, $r(S) \subset Y$) and $r(x) = x$ for each $x \in M$. The assumption that $G(x)$ is finitely metrically closed for each $x \in X$ implies that $Y \cap G(x_i)$ is closed for $i = 1, \dots, n$. Note also that $Y \cap G(x_i) \neq \emptyset$ since, in particular, $y_i \in Y \cap G(x_i)$. However, $\bigcap_{i=1}^n G(x_i) = \emptyset$, so for each $s \in S$ there exists $i_s \in \{1, \dots, n\}$ such that $r(s) \notin Y \cap G\left(x_{i_s}\right)$
Hence $\mathrm{dist}\left(r(s), Y \cap G\left(x_{i_s}\right)\right) > 0$. Therefore if the mapping $f: S \to [0, \infty)$ is defined by setting

$$f(s) := \sum_{i=1}^{n} \mathrm{dist}\left(r(s), Y \cap G(x_i)\right)$$

for each $s \in S$, it must be the case that $f(s) > 0$ for each $s \in S$ and also f is obviously continuous. Now define a (single-valued) mapping $F: S \to S$ by setting

$$F(s) := \frac{1}{f(s)} \sum_{i=1}^{n} \mathrm{dist}\left(r(s), Y \cap G(x_i)\right) y_i$$

for each $s \in S$. Then F is also continuous. Since S is a bounded closed and convex subset of the finite-dimensional space L, by Brouwer's fixed point theorem there exists $s_0 \in S$ such that $F(s_0) = s_0$; i.e.,

$s_0 = F(s_0) = \frac{1}{f(s_0)} \sum_{i=1}^n \mathrm{dist}\left(r(s_0), Y \cap G(x_i)\right) y_i$(2.1)
If $I := \{i_1, \dots, i_k\} = \left\{i \in \{1, \dots, n\}: \mathrm{dist}\left(r(s_0), Y \cap G(x_i)\right) > 0\right\}$(2.2)

then $I \neq \emptyset$, and for each $i \in I$, $r(s_0) \notin Y \cap G(x_i)$. Note that since $r(S) \subseteq Y$ by Proposition 1.1 (3), $r(s_0) \in Y$. (Indeed, $s_0 \in \mathrm{con}\{y_1, \dots, y_n\}$. To see this, note that B is a closed ball centered at a point of M which contains the set $\left\{y_{i_j} : j = 1, \dots, n\right\}$ Since r is nonexpansive and leaves points of M fixed, it follows that $r(y) \in B$. This in turn implies $r(y) \in \mathrm{co}\left(\left\{y_{i_1}, \dots, y_{i_k}\right\}\right)$. Thus, it must be the case that $r(s_0) \notin G(x_i)$ for each $i \in I$, i.e.

$$r(s_0) \notin \bigcup_{i \in I} G(x_i) \quad (2.3)$$

Then by definition of F, we have

$$s_0 = F(s_0) = \frac{1}{f(s_0)} \sum_{j=1}^{k} \mathrm{dist}\left(r(s_0), Y \cap G\left(x_{i_j}\right)\right) y_{i_j} \in \mathrm{conv}\left(\left\{y_{i_1}, \dots, y_{i_k}\right\}\right)$$

This in turn implies $r(s_0) \in \mathrm{co}(\{y_{i_1}, \dots, y_{i_k}\})$. Thus we are able to conclude that $r(s_0) \in \mathrm{co}(\{y_{i_1}, \dots, y_{i_k}\}) \subseteq \bigcup_{j=1}^{k} G\left(x_{i_j}\right)$ and this contradicts Eq. (2.3).

Thus, the family $\{G(x) : x \in X\}$ has the finite intersection property.Therefore, by above result, G is a generalized metric KKM mapping.

Conversely suppose that G is a generalized metric KKM mapping, it follows by above result that the family $\{G(x): x \in X\}$ has the finite intersection property. Rewriting this as $\{G(x) \cap (x_0): x \in X$ and noting $Gx0$ is compact, we have

$$\emptyset \neq \bigcap_{x \in X} G(x) \cap G(x_0) = G(x_0) \bigcap_{x \in X} G(x) = \bigcap_{x \in X} G(x)$$

This completes the proof. ▯

Conclusion

In comparison to the lack of linearity, hyperconvexity provides a very complex metric structure, which leads to a number of surprising and appealing discoveries in a variety of branches of mathematics, including topology, graph theory, multivalued analysis, and fixed point theory. Nonexpansive mappings have traditionally garnered the most attention, since they are at the core of hyperconvex metric space fixed point features. Furthermore, we have highlighted some known consequences of our main results. We are aware of the significance of the well-known Fan-KKM principle in the study of nonlinear analysis, particularly in the study of topology fixed point theory.This research examines various open problems involving the fixed point property in hyperconvex metric spaces. In view of the results, we can say that this technique is a powerful mathematical tool for solving fixed point theory problems. Also, we can use it to obtain approximate solutions to other problems.

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