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Unsteady State Groundwater Flow in Unconfined Aquifer Under Changing Conditions in Water Levels

Manisha M. Kankarej [a]

[a] Assistant Professor, Department of Mathematics & Statistics, Zayed University, Dubai, UAE

Email Address: manisha.kankarej@gmail.com

Abstract

Generalized solution of linearized Boussinesq equation is derived to approximate the nature of ground water flow due to changes in water levels in two ditches bounding an unconfined aquifer. In this research a mathematical model is developed for two flows. In the first case, a sudden change in the water level occurs in the two bounding ditches simultaneously, while in the second case, a sudden change in water level in one ditch is followed by a gradual rise in water level in the other ditch.

Closed form analytical solutions are obtained for the two cases by solving governing equations using eigen value eigen function method. A numerical example problem is presented in which head values are computed at various places in the aquifer at given time intervals and the results plotted.

Keywords : Boussinesq equation, sloping aquifer, unconfined aquifer, ditch

List of symbols

D = Average saturated depth of the aquifer
H = Dimensionless water head
$h(x, t)$ = Water head height measured from sloping bed
h_d = Initial water levels in the drain
h_0 = Water levels in the drain at $x=0$
h_L = Water levels in the drain at $x=L$
$\hat{h}(x, t)$ = Variable water head height measured from horizontal datum
K = Hydraulic conductivity
L = Lateral extant of the unconfined aquifer
q = Flow rate per unit area of the aquifer
S = Specific yield
t = Time
x = Horizontal x axis (space coordinate)
β = Sloping angle measured in radian
$\alpha = \frac{L \tan \beta}{2D}$
τ = Dimensionless time

Introduction

Groundwater level fluctuations in an aquifer can result from surface filtration or changes in ditch water levels. Managing of this limited water resources has been a challenging task. Ditch water level changes can be due to rapid flooding of stream channels during storm periods or sudden release of water from remote sources in the aquifer. On the other hand, field activities involving pumping operations can lead to rapid pressure build up in canals and ditches to affect the nature of flow in the adjoining aquifer. For instance, if the ditch water level changes abruptly a similar situation can be expected to arise in the aquifer. A gradual rise in the ditch water level on the other hand would be accompanied by a much slower movement in the ground water table than in former case. In this context, mathematical model has been considered as a successful tool to determine and conduct this study.

The actual pattern of the groundwater level changes in the aquifer, is of great interest to hydrologists and irrigation engineers engaged in the design of drainage ditches for controlling high water table build up.

Many researchers Hantush (1962a, 1962b), Spiegel (1967), Glover (1974), Gill (1984) have studied the nature of flow in confined and unconfined aquifer due to sudden changes in the channel water levels.

The case of water table fluctuation in unconfined aquifers resulting from surface infiltration were also treated in earlier studies by Maasland (1959), Huntush (1967), Marino (1974), Mustafa (1987), (1987) and Rai and Mangalik (1999). Although these studies provide useful information about groundwater flow system, but subsurface drainage over hillslope were not satisfactorily explained with these results. To be precise in none of these studies, however, was the case of slow or gradual change in water level in the ditches bordering an aquifer treated was discussed.

In such cases approximation of the groundwater flow based on the assumptions that the streamlines are nearly parallel to the sloping bed (Dupuit-Forchheimer assumption) yields more accurate results. Analytic solutions of linearized Boussinesq equation under varying hydrologic conditions are presented by various investigators Upadhyay and Chauhan (2001), Bansal and Das (2015).

In the current research, a problem leading to a practical situation in the field is considered where the ditch water level in one of the ditches in unconfined aquifer bordering an aquifer changes gradually from a certain initial level to a fixed level. Such a change can result from pumping operations during ditch drainage where the water level is initially controlled, for instance by pump action and later by flow system when the flow is fully developed in the aquifer.

In this research two flow cases are considered. Firstly, a situation in which the water level in two ditches change suddenly is investigated. In the second case water level in one ditch rises suddenly while in the other rises gradually. The resulting solution is then obtained and compared in a sloping aquifer and cases of no slope.

Mathematical formulation

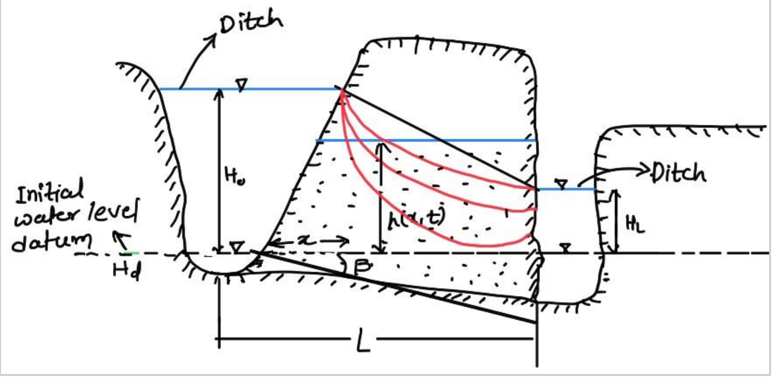


Fig 1. Schematic representation of flow in unconfined aquifer due to rapid and slow changes in ditch water level

R. K. Bansal (2015) studied an unconfined aquifer of lateral extent L overlaying an impermeable bed with an upward slope tan β . With Dupuit-Forchheimer assumptions that the streamline is nearly parallel to the sloping impervious bed, the discharge rate per unit width of the aquifer along x axis can be approximated by following relation Chapman (1980):

$$q = -Kh\, cos^2\beta \frac{\partial}{\partial x}\hat{h} \quad (1)$$

where $\hat{h}$ is the variable water head height measured in vertical direction from horizontal datum and *h(x,t)* is the water head height measured in the vertical direction from the impermeable sloping bed. *K* is the hydraulic conductivity, tan β is the bed slope, *t* = time and *x* = space coordinate. Applying the principle of mass balance across a vertical slice, the equation of subsurface seepage flow over sloping bed is given by:

$$\frac{\partial q}{\partial x} + S\frac{\partial}{\partial x}\hat{h} = 0 \quad (2)$$

where *S* is the specific yield of the aquifer. If the spatial variations in K and S are neglected, then eqns. (1) and (2) imply that

$$Kcos^2\beta\ \frac{\partial}{\partial x}\{h\ \frac{\partial}{\partial x}\hat{h}\} = S\ \ \frac{\partial \hat{h}}{\partial t} \quad (3)$$

Since $\hat{h} = h + x\ tan\ \beta$ after rearranging eqns. (1) and (3) can be written as,

$$q = -Kh\ cos^2\beta\left\{\frac{\partial h}{\partial x} + tan\ \beta\ \right\} \quad (4)$$

$$Kcos^2\beta\ \{\frac{\partial}{\partial x}(h\ \frac{\partial h}{\partial x}) + tan\ \beta\ (\frac{\partial h}{\partial x})\}\ = S\ \frac{\partial h}{\partial t} \quad (5)$$

The problem under treatment is illustrated in Fig 1 showing an unconfined aquifer bounded by two ditches spaced *L* apart. The water levels in two ditches were the same initially at h_d, the datum level.

In the first case, water was suddenly released into both ditches at the same time giving rise to new water level h_0 and h_L in the left hand and right-hand ditches respectively.

In the second case, the rise in water level in the right-hand ditch was gradual while in the left-hand ditch was rapid. Solution of the two cases are sought as follows:

Case I: Sudden Change in ditch water level

Taking the left-hand ditch as the origin with the coordinate system as shown in Fig 1, solution to equation is sought subject to the following initial and boundary conditions:

$$h(x,0)\ =\ h_d, t \le 0 \quad (6a)$$

$$h(0,t)\ = h_0, t > 0 \quad (6b)$$

$$h(L,t)\ = h_L, t > 0 \quad (6c)$$

Eqn. (5) is a second order parabolic partial differential equation, analytic solution of which is not tractable. However approximate analytic solutions can be obtained by taming the non-linearity around some mean saturated depth *D.* Rewriting eqn. (5) as

$$\frac{\partial^2 h}{\partial x^2} + \frac{tan\ \beta}{D}\frac{\partial h}{\partial x}\ = \frac{S}{K\ D\ cos^2\beta}\ \frac{\partial h}{\partial t}, 0 \le\ x\ \le L \quad (7)$$

In fact, this technique invokes linearization of the flow rate *q* around *D*, given by

$$q = -K\ cos^2\beta\left\{D\frac{\partial h}{\partial x} + h\ tan\ \beta\ \right\} \quad (8)$$

The average saturated depth D of the aquifer is approximated using an iterative formula $D = \frac{h_0 + h_t}{2}$, where h_0 is the initial water table height and h_t is the varying water table height at time t, at the end of which D is approximated Marino (1974).

Applying following dimensionless variables

$$H = \frac{h(x,t) - h_d}{L};\ X = \frac{x}{L};\ \tau = \frac{KDcos^2\beta}{SL^2}t;\ \alpha = \frac{L\,tan\beta}{2D}$$

Equation (5) can written as

$$\frac{\partial^2 H}{\partial X^2} + 2\alpha\,\frac{\partial\,H}{\partial X} = \frac{\partial H}{\partial \tau} \qquad (9)$$

The corresponding initial and boundary conditions become

$$H(X, \tau\ =\ 0)\ =\ 0\ ;\ 0\ <\ x\ < L \qquad (10)$$

$$H(0,\ \tau\ ) = \frac{h_0 - h_d}{L};\quad \tau > 0 \qquad (11)$$

$$H(L,\ \tau\ ) = \frac{h_L - h_d}{L}\ ;\quad \tau > 0 \qquad (12)$$

The solution to this boundary value problem is given as:

$$H(X,\tau)\ =\ \frac{1}{L(e^{-2L\alpha}-1)}(2L(e^{-2L\alpha} - 1)(\sum_{n=0}^{\infty}(-\frac{1}{L(L^2\alpha^2+n^2\pi^2)}(((-1)^n(-h_L + h_0)e^{L\alpha} - h_0 + h_d)\pi\ sin(\frac{\pi Xn}{L})(cosh(X\alpha) - sinh(X\alpha))(-sinh(\frac{(L^2\alpha^2+n^2\pi^2)\tau}{L^2}) + cosh(\frac{(L^2\alpha^2+n^2\pi^2)\tau}{L^2}))n))) + (h_d - h_0)e^{-2L\alpha} + (-h_d + h_L)e^{-2X\alpha} + h_0 - h_L) \qquad (13)$$

The above equation can be approximated as below

$$H(X,\tau)\ =\ 2(\sum_{n=0}^{\infty}(-\frac{1}{L(L^2\alpha^2+n^2\pi^2)}(((-1)^n(-h_L + h_0)(1 + L\alpha) - h_0 + h_d)\pi\ sin(\frac{\pi Xn}{L})(cosh(X\alpha) - sinh(X\alpha))(-sinh(\frac{(L^2\alpha^2+n^2\pi^2)\tau}{L^2}) + cosh(\frac{(L^2\alpha^2+n^2\pi^2)\tau}{L^2}))n))) + (h_d - h_0)(1 - 2L\alpha) + (-h_d + h_L)(1 - 2X\alpha) + h_0 - h_L) \qquad (14)$$

In a specific case of an analytical solution, if the impermeable barrier is horizontal $H(X,\tau)$ can be obtained from eqn. (13) by substituting $\beta = 0$ that implies $\alpha = \frac{L\,tan\beta}{2D} = 0\ hence\ \ tan\,\beta\ = 0$. So, above equation takes the form,

$$H(X,\tau)\ =\ 2(\sum_{n=0}^{\infty} - \frac{1}{n\pi L}sin(\frac{\pi Xn}{L})((-1)^n(-h_L + h_0) - h_0 + h_d)(-sinh(\frac{n^2\pi^2\tau}{L^2}) + cosh(\frac{n^2\pi^2\tau}{L^2})) + h_L \qquad (15)$$

Steady State Solution

Steady state solution for eqn. (13) can be obtained by substituting $\frac{\partial H}{\partial \tau} = 0$ on the right side. The solution of the equation for the boundary conditions in eqns. (10), (11) and (12) is obtained as,

$$H(X,\tau)\ =\ \frac{(h_d - h_0)e^{-2L\alpha} + (-h_d + h_L)e^{-2X\alpha} + h_0 - h_L}{L(e^{-2L\alpha}-1)} \qquad (16)$$

For horizontal aquifer, a steady state solution can be obtained from eqn. (16) after expanding the exponential terms $e^{-2\alpha\,L}$ and $e^{-2\alpha\,x}$ in the series form. After, neglecting the greater than 2nd order term and putting $\beta = 0$, above eqn. reduces to

$$H(X,\tau)\ =\ \frac{(h_0 - h_L)}{L} \qquad (17)$$

Case II: Gradual Buildup of ditch water level

Taking the left-hand ditch as the origin with the coordinate system as shown in Fig 1, solution to equation is sought subject to the following initial and boundary conditions:

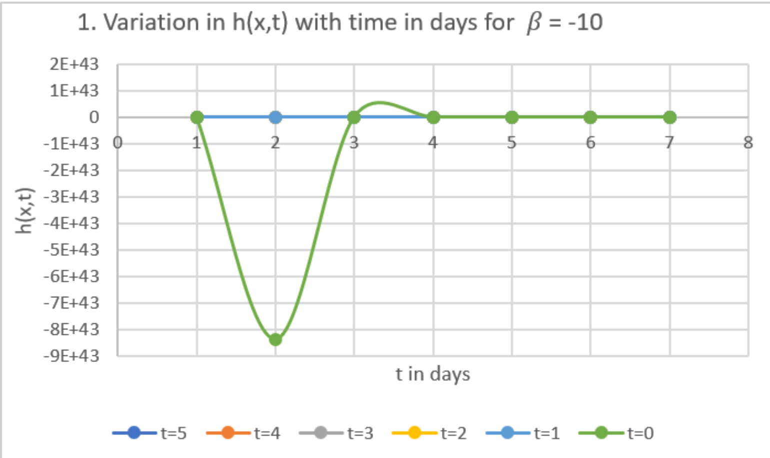
$$h(x,0) \; = \; h_d, t \leq 0 \qquad (18a)$$

$$h(0,t) \; = h_0, t > 0 \qquad (18b)$$

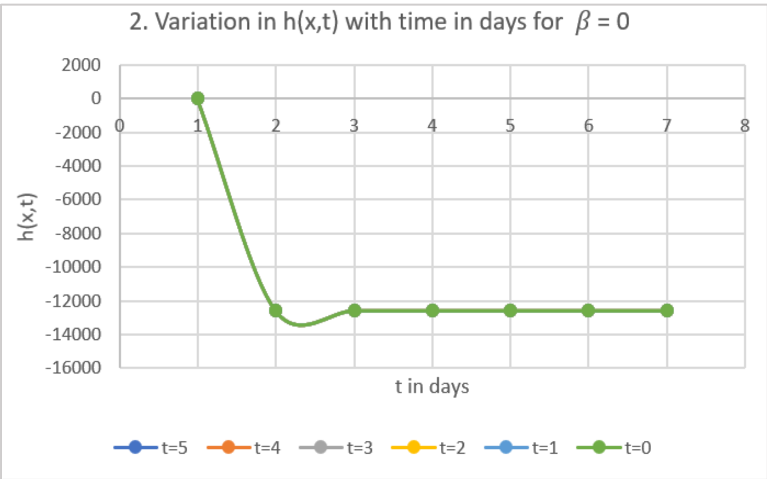
$$h(L,t) \; = h_L(1 - e^{\gamma^2 t})\, erfc(\gamma\sqrt{t}), t > 0 \qquad (18c)$$

Result and discussion

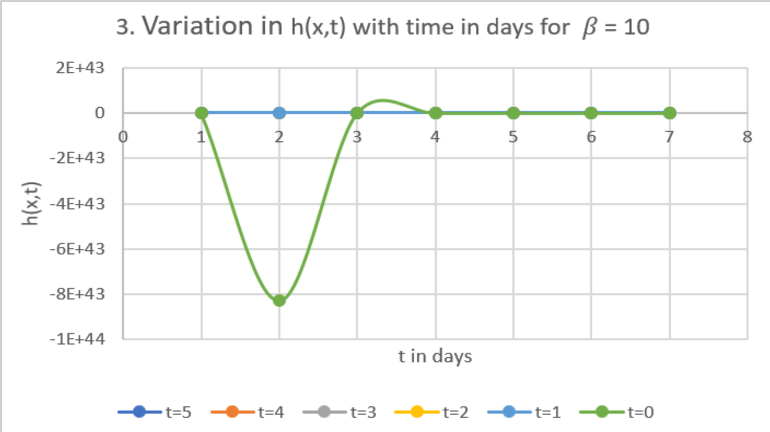
For fixed value of t =1, 2, 3, 4 and 5 days the heads were calculated at distances x = 0 m, x = 10 m, x = 100 m, x = 1000 m. $h_0 = 70$ m, $h_L = 50$ m, $\frac{K}{S} = 12{,}000\ m^2/day, D = 60m.$ The results were plotted as shown in the figure below.



Graph 1. Variation in height of water level with time in days for $\beta = -10$ for rapid water flow.

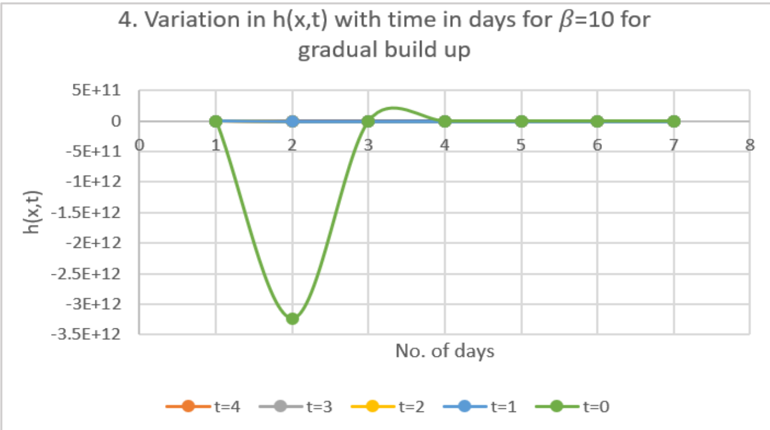


Graph 2. Variation in height of water level with time in days for $\beta = 0$ for rapid water flow.

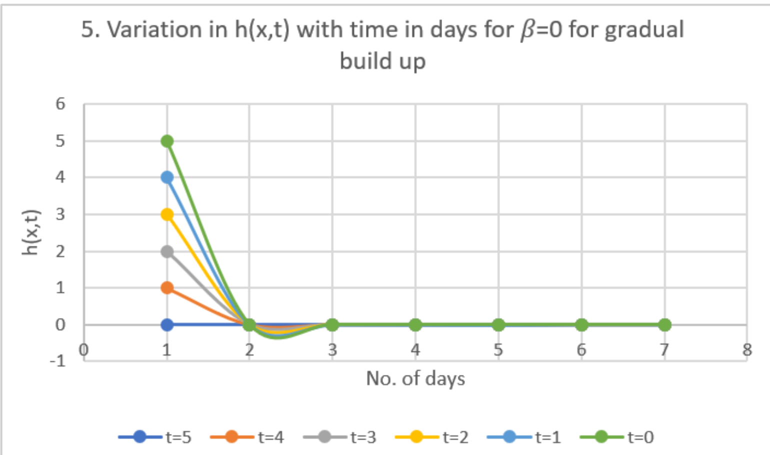


Graph 3. Variation in height of water level with time in days for $\beta = 10$ for rapid water flow.

Case II: For fixed value of t =1, 2, 3, 4 and 5 days the heads were calculated at distances x = 0 m, x = 10 m, x = 100 m, x = 1000 m. $h_0 = 70$ m, $h_L = h_L(1 - e^{\gamma^2 t})\, erfc(\gamma\sqrt{t}), t > 0$ m, $\frac{K}{S} = 12{,}000\ m^2/day, D = 60m.$ With the gradual build-up of the water level we have a delay factor given by, $\gamma = 0.5/(day)^{1/2}$. The results were plotted as shown in the figure below.



Graph 4. Variation in height of water level with time in days for $\beta = 10$ for gradual water flow.



Graph 5. Variation in height of water level with time in days for $\beta = 0$ for gradual water flow.

Conclusion

The head build up is quite fast due to rapid flow of water for the first case with the water profile shown in the figures. Graphs 1 and 3 represents a sloping aquifer. The water level goes down as the water percolates through the surface and after reaching the saturation level it becomes stable as shown in graphs 1 and 3. Whereas in graph 2, with 0 slope, the water gets absorbed and attains stability.

The water profile shown for case II in graph 4 with the sloping aquifer shows that the water level goes down as the water percolates through the surface and after reaching the saturation level it becomes stable. The water level in case II has percolated less as compared to case I as shown in the graphs 1 and 4. Whereas in graph 4, the water gets absorbed and attains stability.

Groundwater level fluctuation in a finite unconfined aquifer resulting from water level changes were investigated in this research. The results were considered for two cases where in the first case we discussed the sudden change in the water level whereas in the second case gradual change of the water level is discussed. The results were discussed for sloping and parallel aquifer and were found in agreement with the results obtained by Mustafa (1987) for a confined aquifer.

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