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Oscillation criterian of first order nonlinear delay differential equation with several deviating arguments.

[1]RAMA RENU, [2]SRIDEVI RAVINDRAN
[1]Associate professor
Department of Mathematics
Quaid-E-Millath Government College for Women,
University of Madras,Chennai,600002, INDIA
[2]Research Scholar
Department of Mathematics
Quaid-E-Millath Government College for Women,
University of Madras,Chennai,600002, INDIA
[1]Email: renurama68@gmail.com,[2]Email: sridevisubramoniam@gmail.com

Abstract

Consider the first order nonlinear delay differential equation with several arguments of the form

$$u'(t)+\sum_{k=1}^{m} q_k(t)\, f_k(u(\delta_k(t)))=0, \qquad t \ge t_0 \ge 0,$$

where the functions $q_k(t), \delta_k(t) \in C([t_{0,}\infty), R),\ \delta_k(t) \le t$ for $t \ge t_0$ and $\lim_{t\to\infty} \delta_k(t)=\infty$ for $1 \le k \le m$. Criterion involving $\limsup$ and $\liminf$ for the oscillation of all solutions of the above equation is obtained. An example illustrating the results is given.

Keywords: *non monotone, nonincreasing, several deviating arguments, delay differential equation.*

1.Introduction

This paper deals with the oscillatory behavior of solution of the first order nonlinear delay differential equation of the form

$$u'(t)+\sum_{k=1}^{m} q_k(t)\, f_k(u(\delta_k(t)))=0, \qquad t \ge t_0 \ge 0, \tag{1.1}$$

where the functions $q_k(t), \delta_k(t) \in C([t_{0,}\infty), R)$ for every $k=1,2,...,m$ and $\delta_k(t)$ are non-monotone or nondecreasing such that

$$\delta_k(t) \le t \text{ for } t \ge t_0 \text{ and } \lim_{t\to\infty} \delta_k(t)=\infty \text{ for } 1 \le k \le m \tag{1.2}$$

and

$$f_k \in C(R,R) \text{ such that } uf_k(u)>0 \text{ for } u \ne 0 \text{ for } 1 \le k \le m. \tag{1.3}$$

In addition, we consider the intial condition for (1.1)

$$u(t)=\varphi(t),\ t\le 0,\ \text{where } \varphi(t):(-\infty,0]\to R \tag{1.4}$$

is a bounded Borel measurable function.

A solution $u(t)$ of (1.1), (1.4) is an absolutely continuous function on $[t_0,\infty)$ satisfying (1.1) for all $t\ge 0$ and (1.4) for all $t\le 0$.

A solution $u(t)$ of (1.1) is oscillatory if it has arbitrary large zeroes. If there exists an eventually positive or an eventually negative solution, the equation is non-oscillatory. An equation is oscillatory if all its solutions are oscillatory.

In the special case for $m=1$, (1.1) reduces to

$$u'(t)+q(t)f(u(\delta(t)))=0 \qquad t\ge t_0\ge 0. \tag{1.5}$$

Recently, there has been a considerable interest in the study of the oscillatory behavior of the following special form of (1.1)

$$u'(t)+q(t)u(\delta(t))=0, \qquad t\ge t_0.$$

In 1987, Ladde, Lakshmikantham and Zhang considered (1-5) with f, q and δ satisfy the following conditions:

i) $\delta(t)\le t$ for $t\ge t_0$ and $\lim_{t\to\infty}\delta(t)=\infty$ and $\delta(t)$ is strictly increasing on R^+,

ii) $q(t)$ are locally integrable and $q(t)\ge 0$,

iii) $f\in C(R,R)$ and $uf(u)>0$ for $u\ne 0$ and $\lim_{u\to 0}\dfrac{u}{f(u)}=P<\infty$.

They proved that if

$$\limsup_{t\to\infty}\int_{\delta(t)}^{t} q(s)ds>P$$

or

$$\liminf_{t\to\infty}\int_{\rho(t)}^{t} q(s)ds>\frac{P}{e},$$

then all solutions of (1.5) are oscillatory.

In 2011, Braverman and Karpuz,[3] considered the linear differential equation

$$u'(t)+q(t)u(\delta(t))=0, \qquad t\ge t_0, \tag{1.6}$$

where q is a function of non-negative real numbers and $\delta(t)$ is a non-monotone of positive real numbers such that $\delta(t)\le t$ for $t\ge t_0$ and $\lim_{t\to\infty}\delta(t)=\infty$. They proved that if

$$\limsup_{t\to\infty} \int_{\rho(t)}^{t} p(s)\exp\left\{\int_{\delta(s)}^{\rho(s)} p(\eta)d\eta\right\}ds > 1$$

where $\rho(t) = \sup_{s\le t} \delta(s)$, $t \ge 0$, then all solution of (1.6) oscillate.

In 2017, Ocalan [11] proved that the following result: Suppose that $\delta(t)$ is not necessarily monotone, $\rho(t) = \sup_{s\le t} \delta(s)$, $t \ge t_0$ and $\lim_{u\to 0} \frac{u}{f(u)} = P$, $0 \le p < \infty$. If

$$\liminf_{t\to\infty} \int_{\rho(t)}^{t} q(s)ds > \frac{P}{e}$$, where $0 \le P < \infty$

or

$$\limsup_{t\to\infty} \int_{\rho(t)}^{t} q(s)ds > 2P$$, where $0 \le P < \infty$,

then all solutions of (1.5) are oscillatory.

Theorem 1.1[7]

Assume that f_k , q_k and δ_k in (1.1) satisfy the following conditions:

i) The condition (1.2) holds and let $\delta_k(t)$ be strictly increasing on R^+,

ii) $q_k(t)$ $(1 \le k \le m)$ are locally integrable and $q_k(t) \ge 0$,

iii) The condition (1.3) holds and let f_k $(1 \le k \le m)$ are nondecreasing functions and

$$\lim_{u\to 0} \frac{u}{f_k(u)} = P_k < \infty .$$

If δ_k are nondecreasing functions for $1 \le k \le m$, and

$$\liminf_{t\to\infty} \int_{\rho(t)}^{t} \sum_{k=1}^{m} q_k(s)ds > \frac{P}{e}$$

or

$$\limsup_{t\to\infty} \int_{\rho(t)}^{t} \sum_{k=1}^{m} q_k(s)ds > P ,$$

where $P = \max_{1\le k\le m} P_k$ and $\delta^*(t) = \max_{1\le k\le m} \delta_k(t)$, then every solution of (1.1) is oscillatory.

Theorem 1.2[7]

Consider the following equation with several arguments of the type

$$u'(t)+q(t)\sum_{k=1}^{m} f(u(\delta_k(t)))=0 \tag{1.7}$$

where $q(t)$ and $\delta_k(t)$ are continuous on $[a,\infty)$, nondecreasing and $\lim_{t\to\infty}\delta_k(t)=\infty$ for $1\le k\le m$. Suppose that $f(u_1,u_2,...,u_m)$ is a continuous function on R^n such that

$$u_1 f(u_1,u_2,...,u_m)>0 \text{ and } u_1u_m>0$$

and

$$P=\lim_{u_k\to 0}\sup\frac{|u_1|^{\alpha_1}...|u_m|^{\alpha_m}}{|f(u_1,u_2,...,u_m)|}<\infty$$

for some nonnegative constants α_k, $1\le k\le m$, with $\sum_{k=1}^{m}\alpha_k=1$. If there is a continuous nondecreasing function $\delta_k(t)\le\delta^*(t)\le t$ for $t\ge a$, $1\le k\le m$ and

$$\liminf_{t\to\infty}\int_{\rho(t)}^{t}\sum_{k=1}^{m}q_k(s)ds>\frac{P}{e},$$

then (1.7) is oscillatory.

The purpose of this paper is to find a new condition for all solutions of (1.1) to be oscillatory when the arguments are not necessarily monotone.

2. Main Results

In this section, we derive new sufficient oscillation conditions, involving limsup and liminf for all solutions of (1.1) under the assumption that $\delta(t)$ is non-monotone function. Set

$$\rho_k(t)=\sup_{t_0\le s\le t}\delta_k(s),\quad t\ge t_0\ge 0 \tag{2.1}$$

and

$$\rho(t)=\max_{1\le k\le m}\rho_k(s). \tag{2.2}$$

Clearly $\rho_k(t)$, $\rho(t)$ are nondecreasing and $\delta_k(t)\le\rho_k(t)\le\rho(t)<t$ for all $t\ge t_0\ge 0$.

Suppose that the function $u(t)$ in (1.1) satisfies the following condition

$$\lim_{|u|\to\infty}\sup\frac{u}{f_k(u)}=P_k,\quad 0\le P_k<\infty. \tag{2.3}$$

Grönwall inequality

Consider the inequality

$$u'(t)+q(t)u(t)\le 0\,, \qquad t\ge t_0\,, \tag{2.4}$$

where $q(t)\ge 0$ and $u(t)\ge 0$. Then we have

$$u(s)\ge u(t)\exp\{\int_s^t q(u)du\}\,, \qquad t_0\le s\le t\,. \tag{2.5}$$

Lemma 2.1[4]

Consider the equation $u'(t)+\sum_{k=1}^{m} q_k(t)\,f_k(u(\delta_k(t)))=0$, $t\ge t_0$. If $q_k(t)\ge 0$, $\delta_k(t)\ge t\ge t_0$, $1\le k\le m$ and if

$$\liminf_{t\to\infty}\int_{\rho(t)}^{t}\sum_{k=1}^{m} q_k(s)ds=l>0$$

then we have

$$\liminf_{t\to\infty}\int_{\delta(t)}^{t}\sum_{k=1}^{m} q_k(s)ds=\liminf_{t\to\infty}\int_{\rho(t)}^{t}\sum_{k=1}^{m} q_k(s)ds=l, \tag{2.6}$$

where $\rho_k(t):=\sup_{t_0\le s\le t}\delta_k(s)$ and $\rho(t)=\max_{1\le k\le m}\rho_k(t)$, $t\ge t_0\ge 0$.

Theorem 2.1

Assume that the hypotheses (1.2), (1.3) and the condition (2.3) hold, if

$$\limsup_{t\to\infty}\int_{\rho(t)}^{t}\sum_{k=1}^{m} q_k(s)\exp\left\{\int_{\delta_k t)}^{\rho_k(s)}\sum_{i=1}^{m} q_i(\eta)d\eta\right\}ds>3P\,, \tag{2.7}$$

where $\delta_k(t)$ are non-monotone or non decreasing and $\delta(t)$ is defined as in (2.1) and $P=\max_{1\le k\le m}\rho_k(t)$, then all the solutions of (1.1) oscillate.

Proof:

Assume for the sake of contradiction, that there exists a non oscillatory solution $u(t)$ of (1.1). Since $-u(t)$ is also a solution of (1.1), whenever $u(t)$ is a solution of (1.1) therefore it is enough to prove the theorem for positive solutions of (1.1). Then, there exists $t_1\ge t_0$ such that $u(t)>0, u(\delta_k(t))>0$ and $u(\rho_k(t))>0$, $1\le k\le m$ for all $t\ge t_1$ .Then, from (1.1) we have

$$u'(t)=-\sum_{k=1}^{m} q_k(t)\,f_k(u(\delta_k(t)))\le 0 \text{ for all } t\ge t_1\,, \tag{2.8}$$

which means that $u(t)$ is an eventually non-increasing function of positive numbers.

Using (2.3) we can choose $t_2\ge t_1$, so large that

$$f_k(u(t)) \geq \frac{1}{3P_k} u(t) \geq \frac{1}{3P} u(t) \text{ for all } t \geq t_2. \tag{2.9}$$

Using (2.9) in (1.1), we have

$$u'(t) + \frac{1}{3P} \sum_{k=1}^{m} q_k(t) u(\delta_k(t)) \leq 0 \text{ for all } t \geq t_2. \tag{2.10}$$

Integrating (2.10) from $\rho(t)$ to t and also using Grönwall's inequality we get

$$u(t) - u(\rho(t)) + \frac{1}{3P} \int_{\rho(t)}^{t} \sum_{k=1}^{m} q_k(s) u(\rho_k(s)) \exp\{\int_{\delta_k(s)}^{\rho_k(s)} \sum_{i=1}^{m} q_i(\eta) d\eta\} ds \leq 0,$$

now using the monotonicity of u we get

$$u(t) - u(\rho(t)) + \frac{1}{3P} u(\rho(t)) \int_{\rho(t)}^{t} \sum_{k=1}^{m} q_k(s) \exp\{\int_{\delta_k(s)}^{\rho_k(s)} \sum_{i=1}^{m} q_i(\eta) d\eta\} ds \leq 0,$$

or

$$-u(\rho(t)) + \frac{1}{3P} u(\rho(t)) \int_{\rho(t)}^{t} \sum_{k=1}^{m} q_k(s) \exp\{\int_{\delta_k(s)}^{\rho_k(s)} \sum_{i=1}^{m} q_i(\eta) d\eta\} ds \leq 0.$$

$$-u(\rho(t)) \left[1 - \frac{1}{3P} \int_{\rho(t)}^{t} \sum_{k=1}^{m} q_k(s) \exp\{\int_{\delta_k(s)}^{\rho_k(s)} \sum_{i=1}^{m} q_i(\eta) d\eta\} ds \right] \leq 0,$$

and hence

$$\int_{\rho(t)}^{t} \sum_{k=1}^{m} q_k(s) \exp\{\int_{\delta_k(s)}^{\rho_k(s)} \sum_{i=1}^{m} q_i(\eta) d\eta\} ds \leq 3P$$

for sufficiently large t. Therefore, we get

$$\limsup_{t\to\infty} \int_{\rho(t)}^{t} \sum_{k=1}^{m} q_k(s) \exp\{\int_{\delta_k(s)}^{\rho_k(s)} \sum_{i=1}^{m} q_i(\eta) d\eta\} ds \leq 3P.$$

This is a contradiction to (2.7). The proof is completed.

Theorem 2.2

Assume that the hypotheses (1.2), (1.3) and the condition (2.3) hold. If $\delta_k(t)$ are non-monotone or non decreasing and if

$$\liminf_{t\to\infty} \int_{\rho(t)}^{t} \sum_{k=1}^{m} q_k(s) \exp\left\{ \int_{\delta_k(s)}^{\rho_k(s)} \sum_{i=1}^{m} q_i(\eta) d\eta \right\} ds > \frac{3P}{e}, \tag{2.11}$$

where $P = \max_{1\leq k\leq m} P_k$ and $\rho(t) = \min_{1\leq k\leq m} \rho_k(t)$, then all solutions of (1.1) oscillate.

Proof:

Suppose to the contrary that (1.1) has a nonoscillatory solution u(t) on $[t_0,\infty)$. Without loss of generality, we can assume that there exists a $t_1 \ge t_0$ such that $u(t) > 0$ and $u(\delta_k(t)) > 0$ on $[t_1,\infty)$.

Thus from (1.1) we have

$$u'(t) = -\sum_{k=1}^{n} q_k(t) f_k(u(\delta_k(t))) \le 0 \text{ for all } t \ge t_1,$$

which means that $u(t)$ is an eventually nonincreasing function of positive numbers.

Case1

Suppose that $P_k > 0$ for $1 \le k \le m$, Then, by (2.3) we can choose $t_2 \ge t_1$, so large that

$$f_k(u(t)) \ge \frac{1}{3P_k} u(t) \ge \frac{1}{3P} u(t) \text{ for all } t \ge t_2. \tag{2.12}$$

Using Grӧnwall inequality in (2.10) , we obtain

$$u'(t) + \frac{1}{3P}\sum_{k=1}^{m} q_k(t) u(\rho_k(t)) \exp\{\int_{\delta_k(s)}^{\rho_k(s)} \sum_{i=1}^{m} q_i(\eta) d\eta\} ds \le 0 \text{ for all } t \ge t_2. \tag{2.13}$$

Using (2.12) and Lemma (2.1), it follows that there exists a constant $d > 0$ such that

$$\int_{\rho(t)}^{t} \sum_{k=1}^{m} q_k(s) \exp\{\int_{\delta_k(s)}^{\rho_k(s)} \sum_{i=1}^{m} q_i(\eta) d\eta\} ds > d > \frac{3P}{e} \text{ for all } t \ge t_3. \tag{2.14}$$

Also, from (2.14) there exists a real number $t^* \in (\rho(t), t)$ for all $t \ge t_3$ such that

$$\int_{\rho(t)}^{t^*} \sum_{k=1}^{m} q_k(s) \exp\{\int_{\delta_k(s)}^{\rho_k(s)} \sum_{i=1}^{m} q_i(\eta) d\eta\} ds > \frac{3P}{2e} \tag{2.15}$$

and

$$\int_{t^*}^{t} \sum_{k=1}^{m} q_k(s) \exp\{\int_{\delta_k(s)}^{\rho_k(s)} \sum_{i=1}^{m} q_i(\eta) d\eta\} ds > \frac{3P}{2e}. \tag{2.16}$$

Integrating (2.13) from $\rho(t)$ to t^*, we get

$$u(t^*) - u(\rho(t)) + \frac{1}{3P}\int_{\rho(t)}^{t^*} \sum_{k=1}^{m} q_k(s) u(\rho_k(s)) \exp\{\int_{\delta_k(s)}^{\rho_k(s)} \sum_{i=1}^{m} q_i(\eta) d\eta\} ds \le 0$$

or

$$u(t^*) - u(\rho(t)) + \frac{u(\rho_k(t^*))}{3P}\int_{\rho(t)}^{t^*} \sum_{k=1}^{m} q_k(s) \exp\{\int_{\delta_k(s)}^{\rho_k(s)} \sum_{i=1}^{m} q_i(\eta) d\eta\} ds \le 0.$$

Using (2.15) in the above inequality, we get

$$-u(\rho(t)) + \frac{u(\rho_k(t^*))}{2e} < 0. \tag{2.17}$$

Similarly, integrating (2.13) from t^* to t and also using (2.16) we get

$$-u(t^*)+\frac{u(\rho_k(t))}{2e}<0. \tag{2.18}$$

Combining (2.17) and (2.18), we get

$$u(t*)>\frac{u(\rho(t))}{2e}>\frac{u(\rho(t^*))}{(2e)^2},$$

and hence we have

$$\frac{u(\rho(t^*))}{u(t*)}<(2e)^2<\infty\ . \tag{2.19}$$

Let $\lambda=\frac{u(\rho(t*))}{u(t*)}$. (2.20)

Then $\lambda\geq 1$ is finite. Now we divide (1.1) by u(t)>0 and integrating from $\rho(t)$ to t we get

$$\int_{\rho(t)}^{t}\frac{u'(s)}{u(s)}ds+\int_{\rho(t)}^{t}\sum_{k=1}^{m}q_k(s)\frac{f_k(u(\delta_k(s)))}{u(s)}ds=0,$$

$$\ln\frac{u(t)}{u(\rho(t))}+\int_{\rho(t)}^{t}\sum_{k=1}^{m}q_k(s)\frac{f_k(u(\delta_k(s)))}{u(\delta_k(s))}\frac{u(\delta_k(s))}{u(s)}ds=0,$$

Then using (2.12) we get

$$\ln\frac{u(t)}{u(\rho(t))}+\frac{1}{3P}\int_{\rho(t)}^{t}q_k(s)\frac{u(\delta_k(s))}{u(s)}ds\leq 0.$$

Since $\delta_k(t)\leq\rho_k(t)\leq\rho(t)$ for $1\leq k\leq m$, we have

$$\ln\frac{u(t)}{u(\rho(t))}+\frac{1}{3P}\int_{\rho(t)}^{t}\sum_{k=1}^{m}q_k(s)\frac{u(\rho_k(s))}{u(s)}\exp\left\{\int_{\delta_k(s)}^{\rho_k(s)}\sum_{i=1}^{m}q_i(\eta)d\eta\right\}ds\leq 0,$$

$$\ln\frac{u(\rho(t))}{u(t)}\geq\frac{1}{3P}\frac{u(\rho_k(\omega))}{u(\omega)}\int_{\rho(t)}^{t}\sum_{k=1}^{m}q_k(s)\exp\left\{\int_{\delta_k(s)}^{\rho_k(s)}\sum_{i=1}^{m}q_i(\eta)d\eta\right\}ds,$$

$$\ln\frac{u(\rho(t))}{u(t)}\geq\frac{1}{e}\frac{u(\rho_k(\omega))}{u(\omega)}, \tag{2.21}$$

where ω is defined by $\rho(t)<\omega<t$.

Using (2.3), (2.13), (2.20) and then taking lim inf on both sides of (2.21), we get

$$\ln \lambda > \frac{\lambda}{e}. \tag{2.22}$$

But (2.22) is not possible since $\ln \mathrm{u} \le \frac{u}{e}$ for all $u > 0$.

Case2

Suppose that $P_k = 0$ for $1 \le k \le m$ and also using the condition (2.3), there exists $t_4 \ge t_3$ such that

$$\frac{u(t)}{f_k(u(t))} < \theta, \quad t \ge t_4$$

and

$$\frac{f_k(u(t))}{u(t)} > \frac{1}{\theta}, \; t \ge t_4 \tag{2.23}$$

where $\theta > 0$ is an arbitrary real number. Thus from (1.1) and (2.23), we have

$$u'(t) + \frac{1}{\theta}\sum_{k=1}^{m} q_k(t)u(\delta_k(t)) < 0.$$

Integrating the above inequality from $\rho(t)$ to t , we get

$$u(t) - u(\rho(t)) + \frac{1}{\theta}\int_{\rho(t)}^{t}\sum_{k=1}^{m} q_k(s)u(\delta_k(s))ds < 0,$$

$$u(t) - u(\rho(t)) + \frac{1}{\theta}\int_{\rho(t)}^{t}\sum_{k=1}^{m} q_k(s)u(\rho_k(s))\exp\{\int_{\delta_k(s)}^{\rho_k(s)}\sum_{i=1}^{m} q_i(\eta)d\eta\}ds < 0,$$

and

$$-u(\rho(t)) + \frac{1}{\theta}u(\rho(t))\int_{\rho(t)}^{t}\sum_{k=1}^{m} q_k(s)\exp\{\int_{\delta_k(s)}^{\rho_k(s)}\sum_{i=1}^{m} q_i(\eta)d\eta\}ds < 0.$$

$$-u(\rho(t))\left(1 - \frac{1}{\theta}\int_{\rho(t)}^{t}\sum_{k=1}^{m} q_k(s)\exp\{\int_{\delta_k(s)}^{\rho_k(s)}\sum_{j=1}^{m} q_k(\eta)d\eta\}ds\right) < 0 \tag{2.24}$$

By using (2.14) and (2.24), we have

$$\frac{d}{\theta} < 1$$

or

$$\theta > d,$$

which is a contradiction to $\lim_{|u|\to 0}\frac{u(t)}{f_k(u(t))} = 0$.

Thus the proof of the theorem is completed.

3 Example

Example 3.1. Consider the equation

$$u'(t)+\frac{1}{10}u(\delta_1(t))\ln(|20+u(\delta_1(t))|)+\frac{2}{10}u(\delta_2(t))\ln(|18+u(\delta_2(t))|)=0\,,\quad t>0\,,$$

where

$$\delta_1(t)=\begin{cases} -t+7k-1, & if\ t\in[4k,4k+1] \\ 3t-9k-5, & if\ t\in[4k+1,4k+2] \\ -2t+11k+5, & if\ t\in[4k+2,4k+3] \\ 3t-9k-10, & if\ t\in[4k+3,4k+4] \end{cases}$$

Figure 1 The graph of $\delta(t)$

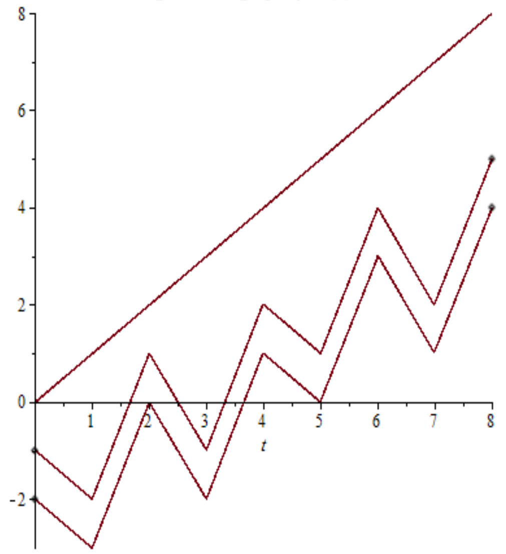
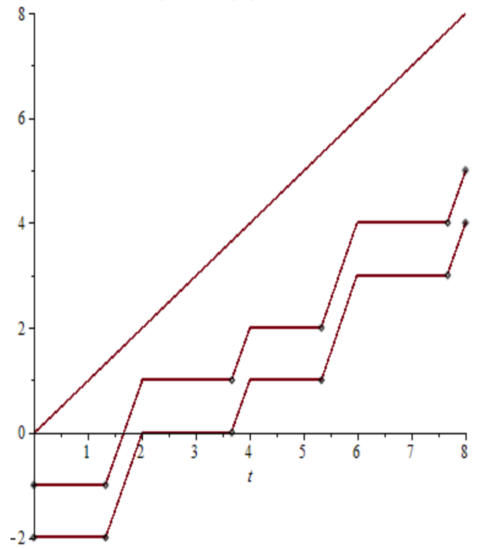


Figure 2 The graph of $\rho(t)$



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By (2.1), we have

$$\rho_1(t)=\sup_{s\le t}\delta_1(s)=\begin{cases} 3k-1, & if\ t\in[4k,4k+\frac{4}{3}] \\ 3t-9k-5, & if\ t\in[4k+\frac{4}{3},4k+2] \\ 3k+1, & if\ t\in[4k+2,4k+\frac{11}{3}] \\ 3t-9k-10, & if\ t\in[4k+\frac{11}{3},4k+4] \end{cases}$$

and $\rho_2(t)=\sup_{t_0\le s\le t}\delta_2(s)=\rho_1(t)-1$, $k\in N_0$ and N_0 is the set of non negative integers.

Therefore

$$\rho(t)=\max_{1\le i\le 2}\left\{\rho_i(t)\right\}=\rho_1(t).$$

If we put $q_1=\dfrac{1}{10}$, $q_2=\dfrac{2}{10}$, $f_1(u)=u(\delta_1(t))\ln(|20+u(\delta_1(t))|)$ and

$f_2(u)=\mathrm{u}(\delta_2(t))\ \ln(|18+u(\delta_2(t))|)$.

Then we have

$$P_1=\limsup_{|u|\to 0}\frac{u(t)}{f_1(u(t))}=\limsup_{|u|\to 0}\frac{u(\delta_1(t))}{u(\delta_1(t))\ln(|20+u(\delta_1(t))|)}=\frac{1}{\ln 20}=0.3338$$

$$P_2=\limsup_{|u|\to 0}\frac{u(t)}{f_2(u(t))}=\limsup_{|u|\to 0}\frac{u(\delta_2(t))}{u(\delta_2(t))\ln(|18+u(\delta_2(t))|)}=\frac{1}{\ln 18}=0.3459$$

$$P'=\max\left\{P_1,P_2\right\}=P_2=\frac{1}{\ln 18}=0.3459.$$

Now at $t=4k+\dfrac{10}{3}$, $k\in N_0$ we have

$$\int_{\rho(t)}^{t}\sum_{k=1}^{2}q_k(s)\exp\left\{\int_{\delta_k(s)}^{\rho_k(s)}\sum_{i=1}^{2}q_i(\eta)d\eta\right\}ds$$

$$=\int_{\rho(t)}^{t}q_1(s)\exp\left\{\int_{\delta_1(s)}^{\rho_1(s)}(q_1(\eta)+q_2(\eta))d\eta\right\}ds+\int_{\rho(t)}^{t}q_2(s)\exp\left\{\int_{\delta_2(s)}^{\rho_2(s)}(q_1(\eta)+q_2(\eta))d\eta\right\}ds$$

$$=\int_{3k+1}^{4k+\frac{10}{3}}\frac{1}{10}\exp\left\{\int_{3s-9k-10}^{3k+1}\frac{3}{10}d\eta\right\}ds+\int_{3k+1}^{4k+\frac{10}{3}}\frac{2}{10}\exp\left\{\int_{3s-9k-11}^{3k}\frac{3}{10}d\eta\right\}ds$$

$$=\int_{3k+1}^{4k+\frac{10}{3}}\frac{1}{10}\exp\left[\frac{3}{10}(12k-3s+11)\right]ds+\int_{3k+1}^{4k+\frac{10}{3}}\frac{2}{10}\exp\left\{\frac{3}{10}(12k-3s+11)\right\}ds$$

$$=\int_{3k+1}^{4k+\frac{10}{3}}\frac{3}{10}\exp\left[\frac{3}{10}(12k-3s+11)\right]ds$$

$$=\frac{1}{3}\left(\exp\frac{3}{10}[3k+8]-\exp\frac{3}{10}[1]\right)>3$$

$$\liminf_{t\to\infty}\int_{\rho(t)}^{t}\sum_{k=1}^{m}q_k(s)\exp\left\{\int_{\delta_k(s)}^{\rho_k(s)}\sum_{i=1}^{m}q_i(u)d\eta\right\}ds>0.38=\frac{3P}{e}$$

$$\limsup_{t\to\infty}\int_{\rho(t)}^{t}\sum_{k=1}^{m}q_k(s)\exp\left\{\int_{\delta_k(s)}^{\rho_k(s)}\sum_{i=1}^{m}q_i(u)d\eta\right\}ds>3>3P$$

All the conditions of Theorem2.1 and Theorem2.2 are satisfied. Hence all solutions of (1.1) are oscillate.

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