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Research Article

On Binary Quadratic Diophantine Equation $11x^2 - 10y^2 = 11$

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ABSTRACT
This paper deals with the problem of obtaining non-zero distinct integer solutions to the non-homogeneous binary quadratic equation represented by the Pell-like equation $11\mathrm{x}^2 - 10\mathrm{y}^2 = 11$. Different sets of integer solutions are presented. Employing the solutions of the above equation, integer solutions for other choices of hyperbolas, parabolas and straight lines are obtained. A special Pythagorean triangle is also determined. The formation of Diophantine 3 – tuples with suitable property is illustrated.

KEYWORDS: Non homogeneous , Binary quadratic, Pell-like equation, hyperbola, parabola, integral solutions, Special numbers.

2010 mathematics subject classification: 11D09

INTRODUCTION

The binary quadratic Diophantine equations (both homogeneous and non homogeneous) are rich in variety $[1-6]$. In $[7-21]$ the binary quadratic non-homogeneous equations representing hyperbolas respectively are studied for their non-zero integral solutions. These results have motivated us to search for infinitely many non-zero integral solutions of an another interesting binary quadratic equation given by $11x^2 - 10y^2 = 11$. The recurrence relations satisfied by the solutions x and y are given. Also a few interesting properties among the solutions are exhibited. A special Pythagorean triangle is also determined. The formation of Diophantine 3 – tuples with suitable property is illustrated.

METHOD OF ANALYSIS

The Diophantine equation representing the binary quadratic equation to be solved for its non-zero distinct integral solution is

$$11x^2 - 10y^2 = 11 \quad (1)$$

Introduce the linear transformation

$$x = X + 10T, y = X + 11T \quad (2)$$

From (1) & (2) we have,

$$X^2 = 110T^2 + 11 \quad (3)$$

whose smallest positive integer solution is

$$X_0 = 11, T_0 = 1$$

To obtain the other solutions of (3), consider the Pell equation

$$X^2 = 110T^2 + 1$$
(4)

whose general solution is given by

$$\widetilde{T_n} = \frac{1}{2\sqrt{110}} g_n, \qquad \widetilde{X_n} = \frac{1}{2} f_n$$

Where $f_n = \left(21 + 2\sqrt{110}\right)^{n+1} + \left(21 - 2\sqrt{110}\right)^{n+1}$

$$g_n = \left(21 + 2\sqrt{110}\right)^{n+1} - \left(21 - 2\sqrt{110}\right)^{n+1}$$

Applying Brahmagupta lemma between (X_0, T_0) and $\left(\widetilde{X_n}, \widetilde{T_n}\right)$, the other integer solutions of (1) are given by

$$x_{n+1} = \frac{21}{2} f_n + \frac{110}{\sqrt{110}} g_n$$

$$y_{n+1} = 11 f_n + \frac{231}{2\sqrt{110}} g_n$$

The recurrence relations satisfied by x and y are given by

$$x_{n+3} - 42x_{n+2} + x_{n+1} = 0$$

$$y_{n+3} - 42y_{n+2} + y_{n+1} = 0$$

Some numerical examples of x and y satisfying (1) are given in the Table:1 below:

Table: 1 Numerical Examples

n	x_{n+1}	y_{n+1}
-1	21	22
0	881	924
1	36981	38786
2	1552321	1628088
3	65160501	68340910

From the above table, we observe some interesting relations among the solutions which are presented below:

1. x_{n+1} is odd & y_{n+1} is even.
2. $x_{2n} \equiv 0 (mod\ 3), n = 0,1,2, \ldots$
3. $y_{2n-1} \equiv 0 (mod\ 4), n = 1,2,3, \ldots.$

Note 1: In addition to (2), one may also consider the linear transformations

$$\mathrm{x} = \mathrm{X} - 10\mathrm{T}, \mathrm{y} = \mathrm{X} - 11\mathrm{T}$$

which lead to a different set of integer solutions to (1).

Note 2: The substitution

$$y = 11T$$
(6)

in (1) leads to

$$x^2 = 110T^2 + 1$$

which is the well known pellian equation whose general solution is given by

$$T_n = \frac{1}{2\sqrt{110}} g_n,\ x_n = \frac{1}{2} f_n$$

In view of (6), one obtains

$$y_n = \frac{11}{2\sqrt{110}} g_n$$

Thus, the above values of x_n, y_n satisfy (1) and these are different from the solutions presented above.

2. Relations among the solutions

- $x_{n+1} - 42x_{n+2} + x_{n+3} = 0$
- $21x_{n+1} - x_{n+2} + 20y_{n+1} = 0$
- $x_{n+1} - 21x_{n+2} + 20y_{n+2} = 0$
- $21x_{n+1} - 881x_{n+2} + 20y_{n+3} = 0$

3. Each of the following expressions represents a nasty number

- $(504x_{2n+2} - 12x_{2n+3} + 12)$
- $\frac{1}{21}(10578x_{2n+2} - 6x_{2n+4} + 252)$
- $(252x_{2n+2} - 240y_{2n+2} + 12)$
- $\frac{1}{21}(10572x_{2n+2} - 240y_{2n+3} + 252)$
- $\frac{1}{881}(443772x_{2n+2} - 240y_{2n+4} + 10572)$

4. Each of the following expressions represents a cubical integer

- $\frac{1}{42}[3526x_{3n+3} - 2x_{3n+5} + 10578x_{n+1} - 6x_{n+3}]$
- $\frac{1}{2}[x_{3n+5} - 1761x_{3n+3} + 3x_{n+3} - 5283x_{n+1}]$
- $84x_{3n+3} - 2x_{3n+4} + 252x_{n+1} - 6x_{n+2}$
- $[21x_{3n+4} - 881x_{3n+3} + 63x_{n+2} - 2643x_{n+1}]$
- $\frac{1}{110}[387860x_{3n+4} - 9240x_{3n+5} + 1163580x_{n+2} - 27720x_{n+3}]$
- $\frac{1}{881}[73962x_{3n+3} - 40y_{3n+5} + 221886x_{n+1} - 120y_{n+3}]$
- $\frac{1}{21}[1762x_{3n+3} - 40y_{3n+4} + 5286x_{n+1} - 120y_{n+2}]$
- $42x_{3n+3} - 40y_{3n+3} + 126x_{n+1} - 120y_{n+1}$

5. Each of the following expressions represents a bi-quadratic integer

- $42x_{4n+4} - 40y_{4n+4} + 168x_{2n+2} - 160y_{2n+2} + 6$
- $1762x_{4n+5} - 1680y_{4n+5} + 7048x_{2n+3} - 6720y_{2n+3} + 6$
- $3522x_{4n+5} - 80y_{4n+6} + 14088x_{2n+3} - 320y_{2n+4} + 6$
- $3526x_{4n+5} - 84x_{4n+6} + 14104x_{2n+3} - 336x_{2n+4} + 6$
- $2x_{4n+5} - 80y_{4n+4} + 8x_{2n+3} - 320y_{2n+2} + 6$
- $\frac{1}{881}[42x_{4n+6} - 70520y_{4n+4} + 168x_{2n+4} - 282080y_{2n+2} + 5286]$
- $\frac{1}{21}[1762x_{4n+4} - 40y_{4n+5} + 7048x_{2n+2} - 160y_{2n+3} + 126]$

6. Each of the following expressions represents a quintic integer

- $42x_{5n+5} - 40y_{5n+5} + 210x_{3n+3} - 200y_{3n+3} + 420x_{n+1} + 400y_{n+1}$
- $1762x_{5n+6} - 1680y_{5n+6} + 8810x_{3n+4} - 8400y_{3n+4} + 17620x_{n+2} - 16800y_{n+2}$
- $\frac{1}{21}[73962x_{5n+6} - 1680y_{5n+7} + 369810x_{3n+4} - 8400y_{3n+5} + 739620x_{n+2} - 16800y_{n+3}]$
- $\frac{1}{231}[462x_{5n+6} - 18480y_{5n+5} + 2310x_{3n+4} - 92400y_{3n+3} + 4620x_{n+2} - 184800y_{n+1}]$
- $\frac{1}{110}[387860x_{5n+6} - 9240x_{5n+7} + 1939300x_{3n+4} - 46200x_{3n+5} + 3878600x_{n+2} - 92400x_{n+3}]$
- $\frac{1}{21}[1763x_{5n+5} - x_{5n+7} + 8815x_{3n+3} - 5x_{3n+5} + 17630x_{n+1} - 10x_{n+3}]$
- $84x_{5n+5} - 2x_{5n+6} + 420x_{3n+3} - 10x_{3n+3} + 840x_{n+1} + 20x_{n+2}$

REMARKABLE OBSERVATIONS:

1. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of straight line which are presented in the table 2 below:

Table 2: Straight Line

S.No	Straight Line	(X, Y)
1.	$21X = Y$	$X = 84x_{n+1} - 2x_{n+2}$ $Y = 1763x_{n+1} - x_{n+3}$
2.	$21X = Y$	$X = 42x_{n+1} - 40y_{n+1}$ $Y = 1763x_{n+1} - x_{n+3}$
3.	$21X = Y$	$X = 42x_{n+1} - 40y_{n+1}$ $Y = 1762x_{n+1} - 40y_{n+2}$
4.	$881X = 21Y$	$X = 1762x_{n+1} - 40y_{n+2}$ $Y = 73962x_{n+1} - 40y_{n+3}$
5.	$110X = 881Y$	$X = 73962x_{n+1} - 40y_{n+3}$ $Y = 387860x_{n+2} - 9240x_{n+3}$
6.	$21X = 10Y$	$X = 387860x_{n+2} - 9240x_{n+3}$ $Y = 462x_{n+2} - 18480y_{n+1}$
7.	$231X = Y$	$X = 1762x_{n+2} - 1680y_{n+2}$ $Y = 462x_{n+2} - 18480y_{n+1}$
8.	$21X = Y$	$X = 1762x_{n+2} - 1680y_{n+2}$ $Y = 73962x_{n+2} - 1680y_{n+3}$
9.	$881X = 21Y$	$X = 73962x_{n+2} - 1680y_{n+3}$ $Y = 42x_{n+3} - 70520y_{n+1}$
10.	$21X = 881Y$	$X = 42x_{n+3} - 70520y_{n+1}$ $Y = 1762x_{n+3} - 70520y_{n+2}$

2. Employing linear combinations among the solutions of (1), one may generateinteger solutions for other choices of hyperbola which are presented in theTable 3 below:

Table: 3 Hyperbolas

S. No	Hyperbola	(X, Y)
1	$110X^2 - Y^2 = 440$	$\begin{pmatrix} 84x_{n+1} - 2x_{n+2}, \\ 21x_{n+2} - 881x_{n+1} \end{pmatrix}$
2	$440X^2 - 441Y^2 = 776160$	$\begin{pmatrix} 1763x_{n+1} - x_{n+3}, \\ x_{n+3} - 1761x_{n+1} \end{pmatrix}$
3	$11X^2 - 4410Y^2 = 19404$	$\begin{pmatrix} 1762x_{n+1} - 40y_{n+2}, \\ 2y_{n+2} - 88x_{n+1} \end{pmatrix}$

4	$X^2 - 110Y^2 = 48400$	$\begin{pmatrix} 387860x_{n+2} - 9240x_{n+3}, \\ 881x_{n+3} - 36981x_{n+2} \end{pmatrix}$
5	$X^2 - 110Y^2 = 213444$	$\begin{pmatrix} 462x_{n+2} - 18480y_{n+1}, \\ 1762y_{n+1} - 44x_{n+2} \end{pmatrix}$
6	$11X^2 - 10Y^2 = 44$	$\begin{pmatrix} 1762x_{n+2} - 1680y_{n+2}, \\ 1762y_{n+2} - 1848x_{n+2} \end{pmatrix}$
7	$4851X^2 - 4410Y^2 = 8557164$	$\begin{pmatrix} 73962x_{n+2} - 1680y_{n+3}, \\ 1762y_{n+3} - 77572x_{n+2} \end{pmatrix}$
8	$11X^2 - 10Y^2 = 34151084$	$\begin{pmatrix} 42x_{n+3} - 70520y_{n+1}, \\ 73962y_{n+1} - 44x_{n+3} \end{pmatrix}$
9	$4851X^2 - 4410Y^2 = 8557164$	$\begin{pmatrix} 1762x_{n+3} - 70520y_{n+2}, \\ 73962y_{n+2} - 1848x_{n+3} \end{pmatrix}$
10	$11X^2 - 10Y^2 = 44$	$\begin{pmatrix} 73962x_{n+3} - 70520y_{n+3}, \\ 73962y_{n+3} - 77572x_{n+3} \end{pmatrix}$
11	$X^2 - 110Y^2 = 484$	$\begin{pmatrix} 21y_{n+2} - 881y_{n+1}, \\ 84y_{n+1} - 2y_{n+2} \end{pmatrix}$
12	$X^2 - 110Y^2 = 853776$	$\begin{pmatrix} 21y_{n+3} - 36981y_{n+1}, \\ 3526y_{n+1} - 2y_{n+3} \end{pmatrix}$

3. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabolas which are presented in the Table: 4 below:

Table: 4 Parabolas

S. No	**Parabola**	$(\boldsymbol{X}, \boldsymbol{Y})$
1	$110X - Y^2 = 220$	$\begin{pmatrix} 84x_{2n+2} - 2x_{2n+3}, \\ 21x_{n+2} - 881x_{n+1} \end{pmatrix}$
2	$11X - 210Y^2 = 462$	$\begin{pmatrix} 1762x_{2n+2} - 40y_{2n+3}, \\ 2y_{n+2} - 88x_{n+1} \end{pmatrix}$
3	$9691X - 10Y^2 = 17075542$	$\begin{pmatrix} 73962x_{2n+2} - 40y_{2n+4}, \\ 42y_{n+3} - 77572x_{n+1} \end{pmatrix}$
4	$X - Y^2 = 220$	$\begin{pmatrix} 387860x_{2n+3} - 9240x_{2n+4}, \\ 881x_{n+3} - 36981x_{n+2} \end{pmatrix}$
5	$21X - 10Y^2 = 9702$	$\begin{pmatrix} 462x_{2n+3} - 18480y_{2n+2}, \\ 1762y_{n+1} - 44x_{n+2} \end{pmatrix}$
6	$11X - 10Y^2 = 22$	$\begin{pmatrix} 1762x_{2n+3} - 1680y_{2n+3}, \\ 1762y_{n+2} - 1848x_{n+2} \end{pmatrix}$
7	$231X - 10Y^2 = 9702$	$\begin{pmatrix} 73962x_{2n+3} - 1680y_{2n+4}, \\ 1762y_{n+3} - 77572x_{n+2} \end{pmatrix}$
8	$9691X - 10Y^2 = 17075542$	$\begin{pmatrix} 42x_{2n+4} - 70520y_{2n+2}, \\ 73962y_{n+1} - 44x_{n+3} \end{pmatrix}$
9	$231X - 10Y^2 = 9702$	$\begin{pmatrix} 1762x_{2n+4} - 70520y_{2n+3}, \\ 73962y_{n+2} - 1848x_{n+3} \end{pmatrix}$
10	$11X - 10Y^2 = 22$	$\begin{pmatrix} 73962x_{2n+4} - 70520y_{2n+4}, \\ 73962y_{n+3} - 77572x_{n+3} \end{pmatrix}$
11	$X - 10Y^2 = 22$	$\begin{pmatrix} 21y_{2n+3} - 881y_{2n+2}, \\ 84y_{n+1} - 2y_{n+2} \end{pmatrix}$
12	$42X - 10Y^2 = 38808$	$\begin{pmatrix} 21y_{2n+4} - 36981y_{2n+2}, \\ 3526y_{n+1} - 2y_{n+3} \end{pmatrix}$

4. Consider $p = x_{n+1} + y_{n+1},\ q = y_{n+1}$ Note that $p > q > 0$ Treat p, q as the generators of the Pythagorean triangle $T(X, Y, Z)$ where $X = 2pq, Y = p^2 - q^2, Z = p^2 + q^2$.

Then the following results are obtained:

$11X - 5Y - 6Z + 11 = 0$

a) $\frac{2A}{P} = x_{n+1}y_{n+1}$, where $A = area$ and $P = perimeter$.

b) $3\left(X - \frac{4A}{P}\right)$ is a nasty number.

c) $X - \frac{4A}{P} + Y$ is written as the sum of two squares.

Formation of sequence of Diophantine 3-tuples

Consider the solution to (1) given by

$$x_0 = 1, y_0 = 22$$

It is observed that

$x_0 y_0 + k^2 + 42k - 21 = (k+21)^2$, a perfect square.

Therefore, the pair (x_0, y_0) represents Diophantine 2-tuples with property $D(k^2 + 42k - 21)$.

If c is the 3rd tuple, then it satisfies the system and double equations,

$$21c + k^2 + 42k - 21 = p^2 \quad (7)$$

$$22c + k^2 + 42k - 21 = q^2 \quad (8)$$

Eliminating c between (7) and (8), we have

$$22p^2 - 21q^2 = k^2 + 42k - 21 \quad (9)$$

Taking

$$p = X + 21T \qquad q = X + 22T \quad (10)$$

in (9) and simplifying, we get

$$X^2 = 21 + 22T^2 + k^2 + 42k - 21$$

which is satisfied by

$$T = 1, X = k + 21$$

In view of (10) and (7), it is seen that

$$c = 2k + 85$$

Note that $(21, 22, 2k + 85)$ represents Diophantine with property $D(k^2 + 42k - 21)$

This property of obtaining sequences of Diophantine 3-tuples with this property $D(k^2 + 42k - 21)$ is illustrated below:

Let M be a 3×3 nmatrix given by

$$M = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix}$$

Now, $(21 \quad 22 \quad 2k+85)M = (21 \quad 2k+85 \quad 4k+190)$

Note that

$$21(2k+85) + k^2 + 42k - 21 = (k+42)^2$$

$$21(4k+190) + k^2 + 42k - 21 = (k+63)^2$$

$$(2k+85)(4k+190) + k^2 + 42k - 21 = (3k+127)^2$$

$\therefore$ The triple $(21 \quad 2k+85 \quad 4k+190)$ represents Diophantine 3 – tuple with the property $D(k^2+42k-21)$. The repletion of the above process leads to sequences of Diophantine 3 – tuples whose general form$(21, c_{s-1}, c_s)$ is given by

$$(21, \quad 21s^2+2(s-1)k+1, \quad 21s^2+2(k+21)s+22), s=1,2,3 \ldots$$

A few numerical illustrations are given in the table below :

Table 5 : Numerical Illusrations

k	$(21. \quad c_0, \quad c_1)$	$(21. \quad c_1, \quad c_2)$	$(21. \quad c_2, \quad c_3)$	$D(k^2+42k-21)$
0	$(21, \quad 22, \quad 85)$	$(21, \quad 85, \quad 190)$	$(21, \quad 190, \quad 337)$	$D(-21)$
1	$(21, \quad 22, \quad 87)$	$(21, \quad 87,. \quad 194)$	$(21, \quad 194, \quad 343)$	$D(22)$
2	$(24, \quad 22, \quad 89)$	$(21, \quad 89, \quad 198)$	$(21, \quad 198, \quad 349)$	$D(67)$

It is noted that the triple $(c_{s-1,} \quad c_s+21 \quad c_{s+1}), s=1,2,3 \ldots$ forms an arithmetic progression.

In a similar way, one may generate sequences of Diophantine 3-tuples with suitable property through the other solutions to (1).

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