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Research Article

Asymptotic Distribution Of The Central Variation Members In The Case Of Random Sampling Volume

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ABSTRACT

This work is devoted to studying asymptotic distributions of the central variation members in random sampling volume numbers. However, this does not imply the independence of the observed values of the aggregate as a whole from the sample size.

Keywords: random value, variation series, sample volume, central members of the variation series, asymptotic distribution.

INTRODUCTION

Let $X_1, X_2, \dots, X_n, \dots$ a sequence of independent random values (r.v.) with general function of distribution (d.f.) $F(x) = P(X_1 < x)$ and - a variation series (v.s.) $\xi_1^{(n)} \le \xi_2^{(n)} \le \dots \le \xi_n^{(n)}$ built according to the r.v. $X_1, X_2, \dots, X_n, \dots$

The relationship $\frac{k}{n}$ is called the rank of a member $\xi_k^{(n)}$. If at $n \to \infty$, $\frac{k}{n} \to \lambda$, $0 \le \lambda \le 1$, then λ is called the limit rank of the sequence $\left\{\xi_k^{(n)}\right\}$. Members $\xi_k^{(n)}$ for which λ a non-zero and one is called the central members of the v.s., and $\xi_k^{(n)}$ members for which the maximum rank $\lambda = 0$ or $\lambda = 1$ is called the extreme members of the v.s.

Variation series is the starting point for many applications, and the concept itself is widely used in mathematical statistics and other fields of knowledge. Therefore, a large number of work is devoted to studying the distribution of members of the variation series. The complete results, completing the theory of the maximum distribution of members of v.s., were obtained in works by B.V.Gnedenko [1], N.V.Smirnov [2], and D.M.Chibisov [3].

In the classical mathematical statistics and the studies of the above authors, the sample size on which the variation series is formed is considered deterministic.

Random sample size appears in statistical tasks of the theory of reliability, theory of mass service, sequential analysis, etc. In this paper, we study the asymptotic distributions of the central members of the variation series when the sample size itself is a random value, i.e., the characteristics of the aggregate considered were observed (due to some circumstances) in the random number of tests. This situation was often found in practice and was more general than a deterministic case where observations were considered non-random. At the same time, the random

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sample size might be independent (call this case "independent scheme"), and in some cases, dependent on the observed values themselves (call this case "dependent scheme").

MATERIALS AND METHODS

Studies of asymptotic distributions of members in the v.s. at random sample size in the "dependent scheme," as opposed to the "independent scheme", were very difficult. These studies were based on the concept of mixing in the sense of A.Reni [4], which has a wide field of application.

RESULTS

Consider the variation series $\xi_1^{(\nu_n)} \le \xi_2^{(\nu_n)} \le \ldots \le \xi_{\nu_n}^{(\nu_n)}$, randomly constructed sample $X_1, X_2, ..., X_{\nu_n}$ of the total population with (d.f.) $F(x)$. Here and on $\{\nu_n\}$ - positive integer sequence random value.

Moreover, a detailed study of the maximum distribution of the central members of the range of variations at the deterministic sample volume seemed to begin with the work of N.V.Smirnov [2]. He showed that the class of possible limit allocations for the appropriately centrally and rationed central members of the v.s. consisted of four different types of allocations.

In work [2] under $\left(\frac{k}{n}-\lambda\right)\sqrt{n} \to 0$, $\quad (0<\lambda<1, n\to\infty)$ (1)

found necessary and sufficient conditions to satisfy the d.f. $F(x)$, that belonged to the area of attraction of a certain marginal law of distribution, i.e., appropriate permanent selection $a_n > 0$ *u* b_n the ratio of $P\left\{\frac{\xi_k^{(n)}-b_n}{a_n} < x\right\} = \Phi_{kn}(a_n x + b_n) \to \Psi(x)$, at all points of contiguity of limit f.d. $\Psi(x)$ $\left(n\to\infty, \frac{k}{n}\to\lambda, 0<\lambda<1\right)$.

If you did not make an additional limit (1) on the rate of decrease to zero $\frac{k}{n}-\lambda$, , the same d.f. $F(x)$ might be belonging to different areas of attraction (see [2], h.1). However, in assumption (1), it was possible to distinguish the areas of attraction, which depended only on the nature of the d.f. $F(x)$.

If condition (1) is fulfilled, it is referred to as "normal λ- gravity" and, accordingly, the areas of normal λ - gravity.

In work [2] it was established that the class of possible limit laws of distributions that might have areas of normal λ - gravity for the appropriately centrally and rationalized central members of the v.s. was limited to the following four types of distributions:

1. $$\Phi_\alpha^{(1)}(x) = \begin{cases} 0, & x \le 0, \\ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{cx^\alpha} e^{-\frac{y^2}{2}} dy, & x>0,\ c>0; \end{cases}$$

2. $$\Phi_\alpha^{(2)}(x) = \begin{cases} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-c|x|^\alpha} e^{-\frac{y^2}{2}} dy, & x \le 0, \\ 1, & x>0, \quad c>0; \end{cases}$$

3. $$\Phi_{\alpha}^{(3)}=\begin{cases}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-c_1|x|^{\alpha}} e^{-\frac{y^2}{2}}dy & ,x\le 0\ ,c_1>0\ , \\ \frac{1}{2}+\frac{1}{\sqrt{2\pi}}\int_{0}^{c_2x^{\alpha}} e^{-\frac{y^2}{2}}dy\ , & x>0\ ,\ c_2>0\ ;\end{cases}$$

4. $$\Phi_{\alpha}^{(4)}(x)=\begin{cases}0 & ,\ x\le -1\ , \\ \frac{1}{2} & ,\ -1<x\le 1\ , \\ 1 & ,\ x>1\ .\end{cases}$$

The parameter α took any positive value.

All of these distribution types were $\Phi(u(x))$, *где* $\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{y^2}{2}}dy$, and functions $u(x)$ had one of the following types:

$$u_1(x)=\begin{cases}-\infty & ,\quad x\le 0\ , \\ cx^{\alpha} & ,\quad x>0\ ,\ c>0\ ;\end{cases}\qquad u_2(x)=\begin{cases}-c|x|^{\alpha} & ,\quad x\le 0\ , \\ +\infty & ,\quad x>0\ ;\end{cases}$$

$$u_3(x)=\begin{cases}-c_1|x|^{\alpha} & ,\ x\le 0\ , \\ c_2x^{\alpha} & ,\ x>0\ ,\ c_1,c_2>0\ ,\end{cases}\qquad u_4(x)=\begin{cases}-\infty & ,\ x\le -1, \\ 0 & ,\ -1<x\le 1, \\ +\infty & ,\ x>1.\end{cases}\qquad (2)$$

In this work, the following lemma, which we would use, was an essential place in the research of work [2].

Lemma [2]. If the $\xi_k^{(n)}$ member's rank numbers (i.e., *k* (left) and *n-k+1* (right) at the same time increased indefinitely, then

$$R_{kn}(x)=\Phi_{kn}(a_nx+b_n)-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{u_n(x)} e^{-\frac{y^2}{2}}dy$$ evenly x aspire to zero. Here

$$\Phi_{kn}(a_nx+b_n)=P\left\{\frac{\xi_k^{(n)}-b_n}{a_n}<x\right\}\quad u\quad u_n(x)=\frac{F(a_nx+b_n)-\lambda}{\sqrt{\lambda(1-\lambda)}}\sqrt{n.}$$

From this lemma directly followed that to perform the ratio $\Phi_{kn}(a_nx+b_n)\to\Psi(x),\quad (n\to\infty)$ was necessary and sufficient to $n\to\infty$

$$u_n(x)=\frac{F(a_nx+b_n)-\lambda}{\sqrt{\lambda(1-\lambda)}}\sqrt{n}\to u(x)$$ (*) where a non-descending function $u(x)$ was defined by an equation $\Psi(x)=\Phi\left(u(x)\right)$.

In this paper, we would present the R- mixing property for the central members of the v.s. and examine the central members' maximum distribution in case a random number of observations forms the v.s.

Let $\{\nu_n\}$ - the sequence of positive integers of r.v. and the sequence of the central members of the v.s. still satisfy the condition of regular $\lambda-$attraction (1).

The following theorems were fair.

Theorem 1. Let, with the proper selection of permanent $a_n > 0$, b_n $\lim_{n\to\infty} P\left\{\frac{\xi_{k(n)}^{(n)} - b_n}{a_n} < x\right\} = \Phi(u(x))$, where $u(x)$ has one possible type (2). Then the sequence of the r.v. $\left\{\frac{\xi_{k(n)}^{(n)} - b_n}{a_n}\right\}$ possesses property R – mixing with the ultimate f.d. $\Phi(u(x))$, i.e., for any event A, с $P(A) > 0$

$$\lim_{n\to\infty} P\left\{\frac{\xi_{k(n)}^{(n)} - b_n}{a_n} < x / A\right\} = \Phi(u(x)).$$

Theorem 2. Let, with the proper selection of permanent $a_n > 0$, b_n with $n \to \infty$ be executed:

A) $P\left\{\frac{\xi_{k(n)}^{(n)} - b_n}{a_n} < x\right\} \to \Phi(u(x))$ и

B) $\frac{\nu_n}{n} \to \nu_0$, where $\nu_0 > 0$ - r.v.

In this case, when $n \to \infty$

C) $P\left\{\frac{\xi_{k(\nu_n)}^{(\nu_n)} - b_n}{a_n} < x\right\} \to \int_0^\infty \Phi\left(u(x)\sqrt{y}\right) dP\{\nu_0 < y\}$, where a function $u(x)$ had one of the possible forms.

The main result of work [5] was the statement C) proved in the assumptions of independence ν_n from $\{X_j\}$ and weak convergence $\left\{\frac{\nu_n}{n}\right\}$ to some positive r.v.

It should be noted here that for proof of the type 2 theorems, in case of arbitrary dependence of the random volume ν_n from the original r.d. X_j condition B), in general, cannot be replaced by a condition of weak convergence of the sequence $\left\{\frac{\nu_n}{n}\right\}$ to the positive r.d. ν_0 (see [6]).

Theorem 2 summarized and clarified the results of work [7], which studied the limit distribution of quantile $\xi_{k(\nu_n)}^{(\nu_n)}$, $\left(k(\nu_n) = [\nu_n p] + 1\right)$ $p-$order (0<p<1) in case of random sample size. The following condition is applied to the sequence

$$\frac{\nu_n}{n} \xrightarrow{p} c > 0, \quad (c = const, \ n \to \infty).$$

Theorem 1 and 2 evidence waws based on the following two lemmas, which were of self-interest.

Let for each $n = 1, 2, \ldots$ and fixed x $I_{nk} -$ performance indicator $\{X_k < a_n x + b_n\}$, i.e.

$I_{nk}=\begin{cases}0, & если \quad X_k \ge a_n x+b_n \ , \\ 1, & если \quad X_k < a_n x+b_n \ ,\end{cases}$ where $a_n>0$, b_n- some sequence of valid integers

Let's $p_n=P\{I_{nk}=1\}=F(a_n x+b_n); \ q_n=1-p_n \ u \ S_{nj}=I_{n1}+I_{n2}+...+I_{nj}$.

Lemma 1. If for a sequence of valid integers $\{k_n\}$, $\lim\limits_{n\to\infty} k_n p_n q_n=\infty$, and sequence of r.v. $\left\{\frac{S_{nk_n}-k_n p_n}{\sqrt{k_n p_n q_n}}\right\}$ possessed the property of *R*- mixing with ultimate f.d. $\Phi(x)$, i.e., for any event *A* c *P(A)>0*

$$\lim_{n\to\infty} P\left\{\frac{S_{nk_n}-k_n p_n}{\sqrt{k_n p_n q_n}}<y/A\right\}=\Phi(y).$$

Lemma 2. If for a sequence of valid integers

$\{k_n\}$, $\lim\limits_{n\to\infty} k_n p_n q_n=\infty \ \ u \ \ \frac{\nu_n}{k_n}\xrightarrow{p}\nu_0>0,$ where ν_0 - r.v., and sequence of r.v. $\left\{\frac{S_{n\nu_n}-\nu_n p_n}{\sqrt{\nu_n p_n q_n}}\right\}$

possessed the property of R-mixing with the ultimate f.d. $\Phi(x)$.

Proof of the Theorem 1

Let *A* – an arbitrary event with a positive probability. According to the ratio $\left\{\xi_k^{(n)}<x\right\}=\left\{S_n(x)\ge k\right\}$, где $S_n(x)=I(X_1<x)+I(X_2<x)+...+I(X_n<x)$ (* *)

and $I(A)-$ performance indicator A we had

$$P\left\{\frac{\xi_{k(n)}^{(n)}-b_n}{a_n}<x/A\right\}=P\{S_n(a_n x+b_n)\ge k(n)/A\}=$$

$$=P\left\{\frac{S_n(y_n)-nF(y_n)}{\sqrt{nF(y_n)(1-F(y_n))}}\ge-\frac{nF(y_n)-k(n)}{\sqrt{nF(y_n)(1-F(y_n))}}/A\right\} \ \ где \ y_n=a_n x+b_n.$$

When performing the limit ratio

$$P\left\{\frac{\xi_{k(n)}^{(n)}-b_n}{a_n}<x\right\}\to\Phi\big(u(x)\big)$$

according to the results of [1]

$\lim\limits_{n\to\infty}\frac{F(y_n)-\lambda}{\sqrt{\lambda(1-\lambda)}}\sqrt{n}=u(x) \ \ u \ \ a_n>0, b_n$ were selected so that, where $n\to\infty$

$F(a_n x+b_n)\to\lambda, \ (0<\lambda<1).$

$n\to\infty$

$\frac{nF(y_n)-k(n)}{\sqrt{nF(y_n)(1-F(y_n))}} \to u(x).$ For this reason, using Lemma 1 we would get

$$\lim_{n\to\infty} P\left\{\frac{\xi_{k(n)}^{(n)}-b_n}{a_n} < x / A\right\} = \lim_{n\to\infty} P\left\{\frac{S_n(y_n)-nF(y_n)}{\sqrt{nF(y_n)(1-F(y_n))}} \geq -\frac{nF(y_n)-k(n)}{\sqrt{nF(y_n)(1-F(y_n))}} / A\right\} =$$

$1-\Phi(-u(x)) = \Phi(u(x)).$

The theorem is proven.

Proof of Theorem 2

Again using the ratio (**), we could write the following:

$$P\left\{\frac{\xi_{k(\nu_n)}^{(\nu_n)}-b_n}{a_n} < x\right\} = P\left\{S_{\nu_n}(a_n x + b_n) \geq k(\nu_n)\right\} =$$

$$= P\left\{\frac{S_{\nu_n}(y_n)-\nu_n F(y_n)}{\sqrt{\nu_n F(y_n)(1-F(y_n))}} \geq -\frac{F(y_n)-\frac{k(\nu_n)}{\nu_n}}{\sqrt{F(y_n)(1-F(y_n))}}\sqrt{\nu_n}\right\} \quad где\ y_n = a_n x + b_n. \qquad (3)$$

As it was noted in the proof of theorem 1, when fulfilling the condition

$$u_n(x) = \frac{F(a_n x + b_n)-\lambda}{\sqrt{\lambda(1-\lambda)}}\sqrt{n} \to u(x)\ ,(n\to\infty).$$

And constants $a_n > 0,\ b_n$ are selected so that for any fixed x from many $\{x: 0 < \Phi(u(x)) < 1\}$ where $n\to\infty$

$$F(a_n x + b_n) \to \lambda, \quad (0<\lambda<1). \qquad (4)$$

Let first $x \in \{x: 0 < \Phi(u(x)) < 1\}$. After simple transformations in (3), we had

$$P\left\{\frac{\xi_{k(\nu_n)}^{(\nu_n)}-b_n}{a_n} < x\right\} =$$

$$= P\left\{\left(\frac{S_{\nu_n}(y_n)-\nu_n F(y_n)}{\sqrt{\nu_n F(y_n)(1-F(y_n))}} + \frac{\left(\frac{k(\nu_n)}{\nu_n}-\lambda\right)\sqrt{\nu_n}}{\sqrt{F(y_n)(1-F(y_n))}}\right)\cdot\frac{1}{\sqrt{\frac{\nu_n}{n}}\cdot\sqrt{\frac{\lambda(1-\lambda)}{F(y_n)(1-F(y_n))}}} \geq -u_n(x)\right\}.$$

This ratio was denoted as (5).

By force (4) and conditions B)

$$\sqrt{\frac{\nu_n}{n}}\cdot\sqrt{\frac{\lambda(1-\lambda)}{F(y_n)(1-F(y_n))}}\xrightarrow{p}\sqrt{\nu_0}\ ,\ (n\to\infty). \qquad (6)$$

It was easy to verify that under condition B) $\nu_n \xrightarrow{p} \infty$, $(n\to\infty)$.

Then applying the theorem 1.2.2 [10] on the similarity to the sequence

$$\eta_n=\left(\frac{k(n)}{n}-\lambda\right)\sqrt{n}$$

taking into account the ratio (1) and (4), we get

$$\frac{\left(\frac{k(\nu_n)}{\nu_n}-\lambda\right)\sqrt{\nu_n}}{\sqrt{F(y_n)(1-F(y_n))}}\xrightarrow{p}0\ ,\ \ (n\to\infty). \qquad (7)$$

Using Lemma 2 and properties of *R-mixing* sequences, having combined ratios (5),(6), and (7), we got $n\to\infty$

$$P\left\{\frac{\xi_{k(\nu_n)}^{(\nu_n)}-b_n}{a_n}<x\right\}\to P\left\{\xi\cdot\frac{1}{\sqrt{\nu_0}}\ge -u(x)\right\}, \qquad (8)$$

Where r.v. $\xi -$ has a standard normal distribution.

Due to the observation that the sequence of the r.v. possessing the property R-mixing with some limit d.f. in the limit was independent of any r.v. in relation (8) had

$$P\left\{\xi\cdot\frac{1}{\sqrt{\nu_0}}\ge -u(x)\right\}=P\left\{\xi<u(x)\sqrt{\nu_0}\right\}=\int_0^\infty\Phi\left(u(x)\sqrt{y}\right)dP\left\{\nu_0<y\right\}.$$

Now, let it be $x\in\left\{x:\Phi\left(u(x)\right)=0\right\}$ *или* $x\in\left\{x:\Phi\left(u(x)\right)=1\right\}$.

It was enough to consider the first case when $x\in\left\{x:\Phi\left(u(x)\right)=0\right\}$ where the second case was proved similar to the first.

In the case under consideration $u(x)=-\infty$ and from the ratio (*) it followed that $n\to\infty$

$$\left(F(a_nx+b_n)-\lambda\right)\sqrt{n}\to-\infty. \qquad (9)$$

Then in the ratio (3) we had

$$\frac{F(y_n)-\frac{k(\nu_n)}{\nu_n}}{\sqrt{F(y_n)(1-F(y_n))}}\xrightarrow{p}-\infty,\quad (n\to\infty) \qquad (10)$$

So as

$$\frac{F(y_n)-\frac{k(\nu_n)}{\nu_n}}{\sqrt{F(y_n)(1-F(y_n))}}\sqrt{\nu_n}=\frac{\left(F(y_n)-\lambda\right)\sqrt{n}}{\sqrt{F(y_n)(1-F(y_n))}}\sqrt{\frac{\nu_n}{n}}+\frac{\left(\lambda-\frac{k(\nu_n)}{\nu_n}\right)\sqrt{\nu_n}}{\sqrt{F(y_n)(1-F(y_n))}}\ ,$$

Where effective (9), condition (B) $\frac{\left(F(y_n)-\lambda\right)\sqrt{n}}{\sqrt{F(y_n)(1-F(y_n))}}\sqrt{\frac{\nu_n}{n}} \xrightarrow{p} -\infty$, $(n\to\infty)$ and according to (1) and theorems 1.2.2 [10]

$$\frac{\left(\lambda-\frac{k(\nu_n)}{\nu_n}\right)\sqrt{\nu_n}}{\sqrt{F(y_n)(1-F(y_n))}} \xrightarrow{p} 0, \quad (n\to\infty).$$

Of (10) with Lemma 2 for $x\in\{x:\Phi(u(x))=0\}$ when $n\to\infty$ we would get

$$P\left\{\frac{\xi_{k(\nu_n)}^{(\nu_n)}-b_n}{a_n}<x\right\}\to 0.$$

Similar reasoning for $x\in\{x:\Phi(u(x))=1\}$ had when $n\to\infty$

$$P\left\{\frac{\xi_{k(\nu_n)}^{(\nu_n)}-b_n}{a_n}<x\right\}\to 1.$$

Having collected all the cases considered, we would be convinced of the fairness of the ratio

$$P\left\{\frac{\xi_{k(\nu_n)}^{(\nu_n)}-b_n}{a_n}<x\right\}\to \int_0^\infty \Phi\left(u(x)\sqrt{y}\right)dP\{\nu_0<y\}.$$

Theorem proved. ■

DISCUSSION

Following B.Gnedenko and H.Fahim's [8] theorems, in which a limit distribution exists for a deterministic sequence and under appropriate additional conditions, the existence of a limit distribution for sequences with a random index is approved, we should call theorems of transfer.

In the preface to the monograph by V.M.Kruglov, V.Y.Korolev [9], written by B.V.Gnedenko, the importance, and necessity of studies on transfer theories is indicated. Proving in this work, theorem 2 was a generalization and addition of the transfer theorem for the central members of the variation series of a random sample in an "independent scheme" to the level of "dependent scheme."

CONCLUSION

As a result of the study of asymptotic distributions of the central members of the variation series at random sample size for the case of "dependent scheme":

1.We set the properties of mixing the sequence of central members with deterministic and random sample volumes, which was of self-interest.

2. Proved transfer theorem for the central members of the various series in the "dependent scheme," which was more general and peculiar than "independent scheme."

CONFLICT OF INTERESTS AND CONTRIBUTION OF AUTHORS

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