Turkish Online Journal of Qualitative Inquiry (TOJQI)
Volume 12, Issue 7, July 2021: 4776- 4782

Research Article

The Rigidity And Analytical Inflexibility Of Single-Connected Convex Surfaces Related To A Point And A Plane Along The Edge

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ABSTRACT

One of the consequential problems in the theory of infinitesimal bendings of surfaces is the problem of the rigidity of surfaces in various classes of their deformations. The magnitude of this is dictated not only by the internal development of the very theory of infinitesimal bendings of surfaces and its connection with the analytical rigidity of the surface, but also by its applied side since the strength conditions of structures containing circular shells as elements require, as a rule, the geometric mean rigidity shell surface. This paper investigates infinitesimal bendings of convex surfaces, which along with a certain curve on the surface, are fixed simultaneously with respect to a point and a plane. It is proved that such surfaces in the indicated class of deformations enable rigidity not higher than the second order, and, therefore, are analytically non-bendable.

Key-words: bending, simply connected convex surface, rigidity, analytical rigidity, rotation field, bending field

INTRODUCTION

The great interest that manifests itself in the theory of bendings of infinitesimal bendings of surfaces is explained, on one hand, by deep connections with such branches of mathematics as the theory of differential and integral equations, the theory of generalized analytic functions, etc., and on the other hand, by those important applications. This theory was obtained in mechanics, especially in the theory of thin shells, since, as is known, each non-trivlal infinitely small bending of the middle surface corresponds to a momentless non-stressed state of this shell, unloaded by a surface load and vice versa, to each momentless non-stressed state of an unloaded shell corresponds to a rotation field of infinitesimal bending. N.V. Efimov made his great contribution to the development of the infinitesimal bendings theory of higher orders. In his papers [5], [6] overly intriguing results were obtained establishing a connection between the rigidity of various orders and the analytic rigidity of a surface. In particular, he was able to prove that rigidity of the first and second orders implies the analytic inflexibility of the corresponding surface. Important results in this non-correction were obtained in the works [1], [15].

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This paper considers infinitesimal bendings of the second order and the question of the analytic inflexibility of arbitrary simply connected regular convex surfaces bounded by a closed contour.

The results obtained can be formulated as the following theorem:

Theorem. Let Φ be a simply connected regular convex surface that does not contain flat regions; bounded by a regular closed contour Γ that does not contain line segments that are asymptotic lines of the surface Φ. If such a surface is fixed along a line relative to a point and a plane, then it (surface Φ) becomes analytically unbending.

Proof. To prove the theorem, it suffices to prove that the surface in the considered class of deformations has a rigidity of at most second order, since, as is known [6], the analytic inflexibility of the surface follows.

$$\vec{x} = \vec{x}(u,v), \qquad (u,v) \in D \tag{1}$$

regular surface parametrization Φ, a

$$\vec{x} = \vec{x}(u(s), v(s)), \qquad 0 \le s \le S \tag{2}$$

natural curve parameterization Γ.

Let us assume that the surface Φ in the process of infinitesimal deformation of the second order passes into the surface Φ^* [5]:

$$\vec{x}^*(u,v,t) = \vec{x}(u,v) + 2t\,\overset{1}{\vec{z}}(u,v) + 2t^2\,\overset{2}{\vec{z}}(u,v) \tag{3}$$

$\overset{1}{\vec{z}}(u,v)$ and $\overset{2}{\vec{z}}(u,v)$ - some regular vector functions, t - surface deformation parameter Φ.

In order $\overset{1}{\vec{z}}(u,v)$ и $\overset{2}{\vec{z}}(u,v)$ becomes vector fields which determine the infinitesimal second-order bending of the surface Φ, it is necessary and sufficient that the vector functions $\overset{1}{\vec{z}}(u,v)$ $\overset{2}{\vec{z}}(u,v)$ satisfy the system of equations [6]:

$$\left.\begin{array}{ll} (4.1) & \left(d\vec{x}, d\overset{1}{\vec{z}}\right) = 0, \\ (4.2) & \left(d\vec{x}, d\overset{2}{\vec{z}}\right) + d\overset{1}{\vec{z}}^{\,2} = 0 \end{array}\right\} \tag{4}$$

It is known [6] that the vector - functions $\overset{1}{\vec{z}}(u,v)$ and $\overset{2}{\vec{z}}(u,v)$ correspond to such vector functions $\overset{1}{\vec{y}}(u,v)$ and $\overset{2}{\vec{y}}(u,v)$ that the following equalities hold:

$$\left.\begin{array}{ll} (5.1) & d\overset{1}{\vec{z}} = [\overset{1}{\vec{y}}, d\vec{x}], \\ (5.2) & d\overset{2}{\vec{z}} = [\overset{2}{\vec{y}}, d\vec{x}] + [\overset{1}{\vec{y}}, d\overset{1}{\vec{z}}]. \end{array}\right\} \tag{5}$$

The system of vector fields $(\overset{1}{\vec{y}}, \overset{2}{\vec{y}})$ is uniquely determined by the system of vector fields $(\overset{1}{\vec{z}}, \overset{2}{\vec{z}})$. Conversely, if there is a collection of functions $(\overset{1}{\vec{y}}, \overset{2}{\vec{y}})$ that satisfies Eqs. (5), then Eqs. (4) will also hold and, therefore, deformation (3) will be an infinitesimal second-order bending of the surface Φ.

If from system (5) for the surface Φ follows that $\vec{y}=\overrightarrow{const}$, then the surface Φ has a second-order rigidity.

Let Γ edge of the surface Φ and Φ are an arbitrary point. Then, it is said that the surface Φ is fixed along the Γ curve relative to P point, if in the process of deformation of the surface the distances from the points of the Γ curve to the point P do not change. Further, we will say that the Φ surface is fixed along the Γ curve relative to the ω plane if the class of admissible Φ surface deformations is limited by the condition of stationarity of the distances from the points of the Γ curve to the ω plane.

In order vector functions $\overset{1}{\vec{z}}(u,v)$ and $\overset{1}{\vec{z}}(u,v)$ satisfying equations (4.1) and (4.2) become bending fields of a Φ surface fixed along a Γ curve simultaneously relative to a P point and a P plane, it is necessary and precisely for the following conditions to be satisfied along the curve [4]:

$$\left.\begin{aligned}&(6.1)\quad \left(\overset{1}{\vec{z}}(u(s),v(s,)),\vec{x}(u(s),v(s))-\vec{r}_p\right)_{|\Gamma}=0,\\&(6.2)\quad \left(\overset{1}{\vec{z}}(u(s),v(s,)),\vec{m}\right)_{|\Gamma}=0,\end{aligned}\right\},0\le s\le S \qquad (6)$$

$$\left.\begin{aligned}&(7.1)\quad \left(\overset{2}{\vec{z}}(u(s),v(s,)),\vec{x}(u(s),v(s)-\vec{r}_p)+\overset{1}{\vec{z}}{}^2(u(s),v(s))\right)_{|\Gamma}=0,\\&(7.2)\quad \left(\overset{2}{\vec{z}}(u(s),v(s,)),\vec{m}\right)_{|\Gamma}=0,\end{aligned}\right\},0\le s\le S \qquad (7)$$

$\vec{r}_p$ - point radius vector P; $\vec{m}$ - unit vector perpendicular to plane ω.

Since vectors $(\vec{x}(u(s),v(s))-\vec{r}_p)$ and $\vec{m}$ cannot be collinear at any interval of s variation, the boundary conditions (6) can be represented in this form:

$$\overset{1}{\vec{z}}(u(s),v(s,))=\lambda(s)[\vec{m},\vec{x}(u(s),v(s))-\vec{r}_p] \qquad (8)$$

$\lambda(s)$ - an arbitrary function that characterizes the amount of displacement of the points of the curve Γ in the process of Φ surface deformations.

Differentiating equality (8) with respect to s we enable

$$\frac{d\,\overset{1}{\vec{z}}(u(s),v(s,))}{ds}=\frac{d\lambda(s)}{ds}[\vec{m},\vec{x}(u(s),v(s))-\vec{r}_p]+\lambda(s)[\vec{m},\vec{t}(s)] \qquad (9)$$

$\vec{t}(s)$ is the unit vector of the tangent vector of the curve $\vec{x}=\vec{x}(u(s),v(s))$.

If we multiply both sides of this equality scalarly by a vector $\vec{t}(s)$ and take into account equality (4.1), then we obtain

$$\frac{d\lambda(s)}{ds}\left(\vec{m},\vec{x}\left(u(s),v(s)\right)-\vec{r},\vec{t}(s)\right)=0. \qquad (10)$$

Depending on the shape of the contour Γ, we will consider two possible cases:

First case. The curve Γ does not contain plane segments lying in planes that are perpendicular to the ω plane and pass through a P point.

Second case. A curve Γ contains plane segments lying in planes that are perpendicular to the ω plane and pass through a P point.

Item 1. The first case is considered.

Here $\left(\vec{m}, \vec{x}\left(u(s), v(s)\right) - \vec{r}, \vec{t}(s)\right) \neq 0$, therefore, equality (10) implies that $\frac{d\lambda(s)}{ds} = 0$, i. e $\lambda(s) = \lambda_0 = const$

Then equality (9) takes the form:

$$\frac{d\overset{1}{\vec{z}}\left(u(s), v(s)\right)}{ds} = \lambda_0 \left[\vec{m}, \vec{t}(s)\right]. \tag{11}$$

So, for a vector - function $\overset{1}{\vec{z}}(u,v)$ along a contour Γ, we get the same restriction as in [4].

Further, using the Blaschke formula [15]:

$$2\iint_{\Phi}\left(\overset{1}{\beta}\overset{1}{\gamma} - \overset{1}{\alpha}{}^2\right)\left(\vec{x}, \vec{x}_u, \vec{x}_v\right)dudv = \oint_{\Gamma}\left(\vec{x}, \overset{1}{\vec{y}}, d\overset{1}{\vec{y}}\right), \tag{12}$$

$\overset{1}{\alpha}(u,v)$, $\overset{1}{\beta}(u,v)$, $\overset{1}{\gamma}(u,v)$ - expansion coefficients of the rotation field $\overset{1}{\vec{y}}(u,v)$ in vectors $\vec{x}_u, \vec{x}_v$:

$$\left.\begin{aligned} \overset{1}{\vec{y}}_u(u,v) &= \overset{1}{\alpha}(u,v)\vec{x}_u(u,v) - \overset{1}{\beta}(u,v)\vec{x}_v(u,v), \\ \overset{1}{\vec{y}}_v(u,v) &= \overset{1}{\gamma}(u,v)\vec{x}_u(u,v) - \overset{1}{\alpha}(u,v)\vec{x}_v(u,v), \end{aligned}\right\}$$

while scalar functions $\overset{1}{\alpha}(u,v)$, $\overset{1}{\beta}(u,v)$, $\overset{1}{\gamma}(u,v)$ are solutions of the system:

$$\left.\begin{aligned} & L\overset{1}{\gamma} - 2M\overset{1}{\alpha} + N\overset{1}{\beta} = 0, \\ & \overset{1}{\alpha}_v - \overset{1}{\gamma}_u = \Gamma_{11}^1\overset{1}{\gamma} - 2\Gamma_{12}^1\overset{1}{\alpha} + \Gamma_{22}^1\overset{1}{\beta}, \\ & \overset{1}{\alpha}_u - \overset{1}{\beta}_v = \Gamma_{11}^2\overset{1}{\gamma} - 2\Gamma_{12}^2\overset{1}{\alpha} + \Gamma_{22}^2\overset{1}{\beta}, \end{aligned}\right\}$$

$L(u,v)$, $M(u,v)$, $N(u,v)$ - coefficients of the second quadratic form; Γ_{ij}^k - Christoffel symbols of the second kind of surface.

We come to the conclusion that in the case when the contour Γ does not contain plane segments, the planes of which pass through the point P and are perpendicular to the plane ω, Φ surface in the indicated class of deformations has first-order stiffness, and therefore [6], is analytically unbendable.

Item 2. The second case is considered.

The case when Γ contains line segments lying in planes that pass through a P point and are perpendicular to the plane ω. Let us denote the union of all such segments by Γ'. Then, Γ' along is equal to $\left(\vec{x}, \vec{m}, \vec{t}\right)_{|\Gamma'} = 0$.

Two cases are also possible here:

- **a)** all segments from the set Γ' lie in the same plane, which we denote by π;
- **b)** some segments from (or all of them) lie in different planes.

Let's consider case a):

Since, $\frac{d\overset{1}{\vec{z}}}{ds}\Big|_{\Gamma} = \left[\overset{1}{\vec{y}}, \vec{t}\right]$, from equation (9) it follows that along Γ' $\left(\overset{1}{\vec{y}}, \vec{t}, \vec{m}\right)_{|\Gamma'} = 0$ where is the $\vec{t}(s)$ unit vector of the tangent to Γ'. It follows that the vectors $\overset{1}{\vec{y}}_{|\Gamma'} = 0$ and $\frac{d\overset{1}{\vec{y}}}{ds}\Big|_{\Gamma'}$ are coplanar to the fixed plane π, which is determined by the vectors $\vec{m}$ and $\vec{t}(s)$. Therefore, the following equality Γ' holds along the segments:

$$\left[\overset{1}{\vec{y}}, \frac{d\overset{1}{\vec{y}}}{ds}\right]_{|\Gamma'} = \varphi(s)\left(\vec{m}, \vec{t}(s)\right)_{|\Gamma'} \qquad (13)$$

$\varphi(s)$ - an arbitrary continuous function of s.

If now we select O point that belongs to π plane as the origin of coordinates, then along the segments Γ' : $\left(\vec{x}, \overset{1}{\vec{y}}, d\overset{1}{\vec{y}}\right) = 0$ But, since, along the segments belonging to $\Gamma \setminus \Gamma'$: $\left(\vec{x}, \overset{1}{\vec{y}}, d\overset{1}{\vec{y}}\right)_{\Gamma\setminus\Gamma'} = 0$ regardless of the choice of the origin, now equality will be performed along Γ the entire contour $\oint_{\Gamma}\left(\vec{x}, \overset{1}{\vec{y}}, d\overset{1}{\vec{y}}\right) = 0$.

Since the point O (origin of coordinates) can be chosen in π plane so that it is inside the convex hull defined by Φ surface, taking into account Blaschke's identities (12), it can be concluded that the surface Φ in the considered class of deformations has first-order rigidity and, therefore, is analytically unbendable.

Let's consider case b):

When the contour Γ contains plane segments, the planes of which are perpendicular to the plane of attachment ω and pass through P point, and at the same time not all planes of such segments coincide with each other. The union of all such flat contour Γ segments is denoted by Γ''. Let us prove that in this case the surface Φ in the considered class of deformations has a rigidity of at most second order. Let g'' be one of the segments belonging to Γ''. We choose a rectangular Cartesian coordinate system, aligning the origin with an arbitrary point O located inside a closed convex hull $\overline{\Phi}$ defined by the surface $\overline{\Phi}$, and as coordinate vectors $\vec{i}, \vec{j}, \vec{k} = \left[\vec{i}, \vec{j}\right]$ where $\vec{i} = \vec{m}$ while $\vec{j}$ is a unit vector parallel to the straight line along which the planes of the curve g'' and intersect g''.

In the so chosen coordinate system, the radius is the vector $\vec{x}(u(s), v(s))$ of an arbitrary point of the curve g'' and the radius is the vector point P can be represented as:

$$\vec{x} = \rho_1(s)\vec{i} + \rho_2(s)\vec{j} + \rho_3(s)\vec{k}, \qquad (14)$$

$$\vec{r}_p = p_1\vec{i} + p_2\vec{j} + p_3\vec{k}, \qquad (15)$$

p_1, p_2, p_3 - constant numbers. Then, the system of equations (7) can be written as follows:

$$\left[\left(\overset{2}{\vec{z}}, \vec{j}\right) + \frac{(\overset{1}{\vec{z}}, \overset{1}{\vec{z}})}{\rho_2(s) - p_2}\right]_{|g''} = 0,$$

$$\left(\overset{2}{\vec{z}}, \vec{i}\right)_{|g''} = 0,$$

$\rho_2(s) - p_2 \neq 0$ - differentiating these equalities along the segment g'', respectively, we enable

$$\left[\left(\vec{j}, d\overset{2}{\vec{z}}\right) + d\left(\frac{(\overset{1}{\vec{z}}, \overset{1}{\vec{z}})}{\rho_2(s) - p_2}\right)\right]_{|g''} = 0 \qquad (16)$$

$$\left(i, d\overset{2}{\vec{z}}\right)_{|g''} = 0. \qquad (17)$$

If we add equalities (16) and (17), having previously multiplied them by $d(\rho_2(s))$ and $d(\rho_2(s))$, respectively, and taking into account (4.2), (14), and (15), then we obtain

$$\left[-\left(d\overset{1}{\vec{z}}, d\overset{1}{\vec{z}}\right) + d(\rho_2(s) - p_2) d\left(\frac{(\overset{1}{\vec{z}}, \overset{1}{\vec{z}})}{\rho_2(s) - p_2}\right)\right]_{|g''} = 0.$$

Due to this we find:

$$\overset{1}{\vec{z}}_{|g''} = (\rho_2(s) - p_2)\vec{c}, \qquad (18)$$

$\vec{c}$ - arbitrary constant vector.

However, from equality (8), taking into account expressions (14) and (15), we enbale

$$\overset{1}{\vec{z}}_{|g''} = \lambda(s)(\rho_2(s) - p_2)\left[\vec{i}, \vec{j}\right]. \qquad (19)$$

Then, by comparing equalities (18) and (19), we find

$$\overset{1}{\vec{z}}_{|g''} = c_0(\rho_2(s) - p_2)\left[\vec{i}; \vec{j}\right],$$

c_0 - arbitrary constant.

So, for a vector - function $\overset{1}{\vec{z}}(u,v)$ along a segment g'', we got the same restriction as in Item 1. Further, using the Blaschke formula (12), we come to the conclusion that on the entire surface Φ $\overset{1}{\vec{y}}(u,v) = \overrightarrow{const}$ which means that the surface Φ in the case under consideration has a second-order rigidity and, therefore, is analytically non-bendable.

CONCLUSION

In conclusion, we have proved that simply connected regular convex surfaces in the indicated class of deformations enable a rigidity of no higher than the second order, and, therefore, are analytically non-bendable

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