Turkish Online Journal of Qualitative Inquiry (TOJQI)
Volume 12, Issue 8, July 2021: 3470 – 3476

L^1-Convergence of Modified Trignometric Sum Under Some Classes of Coefficients and Its Generalization

Sakshi[1] , Karanvir Singh[2]
[1]Research Scholar,
Department of Mathematics , Maharaja Ranjit Singh Punjab
Technical University, Bathinda, Punjab, India
sakshi.bfcmt@gmail.com
[2] Research Supervisor
Department of Mathematics , Maharaja Ranjit Singh Punjab
Technical University, Bathinda, Punjab, India
karanvir@mrsptu.ac.in

Abstract: In this paper we introduce new modified cosine sums and thenusing the sums we study the necessary and sufficient condition for L^1 - Convergence of trigonometric cosine series underclass S & C. Also we do generalization of this modified sum and proved the necessary and sufficient condition for the L^1 convergence of this generalized sum.

2010 Mathematics subject classification: 42A20, 42A32.

Key words and phrases: L^1-convergence ,Fezer Kernel, modifiedcosine sums.

1. Introduction

Let

(1.1) $\frac{a0}{2} + \sum_{k=1}^{\infty} a_k \cos kx$

be the cosine series.and the partial sum of (1.1) be denoted by $S_n(x)$ and $f(x) = \lim_{n\to\infty} S_n(x)$.

The integrabilty and L^1 convergence of trignometric series is studied by different authors time to time and if we go back to the history of L^1 convergence of above trignometric series, very first time, it was studied by Young[19] and Kolmogorov[10] and proving the integrability of cosine series by taking the classes of convex and quasi- convex sequences respectively.

Definition 1. [17,18] : A sequence a_k is said to belong to class S if$a_k = o(1)$,$k \to \infty$ and there exist a sequence A_k such that (a) $A_k \downarrow 0, k \to \infty$(b)$\sum_{k=0}^{\infty} A_k < \infty$(c) $|a_k| \le A_k$

Definition 2. A null sequence ak belongs to the class Cr; r = 0; 1; 2; 3; …….. , if for every $\varepsilon > 0$; $\Delta a > 0$ such that $\int_0^{\pi} | \sum_{k=n}^{\infty} \Delta a_k D_k^r(x) | dx < \varepsilon$, for all n. where

$D_k^r(x)$ is the r-th derivative of Drichlet Kernel. Whenr = 0 we denoteCr = Ci.e. A null sequence an belongs to the class C if for every $\varepsilon > 0$, there existsa $\Delta(\varepsilon) > 0$, independent of n, such that

$$\int_0^{\delta} | \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) | dx < \varepsilon, \text{ for all } n \geq 0$$

Many authors by studying the behaviour of L1 convergence of above said trignometric series proved the same necessary and sufficient condition , under different classes of coefficients, which is as follows:

$$a_n \log n = o(1);\ n \to \infty \text{ iff } \|f - S_n\| = o(1);\ n \to \infty \qquad (*)$$

So many modifications are done by many authors while proving the L1 convergence of cosine series . although they introduced so many classes to prove the result(*).The famous authors like Rees and C.V. Stanojevic [14], Kumari and Ram [12], K. Kaur, Bhatia and Ram [9] , J. Kaur [8], Braha [3] and Krasniqi [11], proved the necessary and sufficient condition for L^1 convergence by introducing modified sine and cosine sums.

Rees and Stanojevi´c [14] introduced following modified cosine sums

$$g_n(x) = \frac{1}{2}\sum_{k=0}^{\infty} \Delta a_k + \sum_{k=1}^{n} \sum_{j=k}^{n} (\Delta a_j) \cos kx$$

Following are the results which are proved by Garett and Stanojevic' by considering the class C [6] of bounded variation Garett and Stanojevic' proved the following Result :

Theorem A. If a_k belong to the class C and is of bounded variation , then $\|f - g_n\| = o(1)$; $n \to \infty$

By considering the class S , Ram proved the folowing theorem:

Kumari and Ram introduced modified cosine sums as

$$h_n(x) = \frac{a0}{2} + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta \left(\frac{aj}{j}\right) k \cos kx$$

and studied their L1-convergence under the condition that coefficient sequence ak belong to the class S.

Garret and Stanojevic [1] have introduced modified cosine sums

$$f_n(x) = \frac{1}{2}\sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n} \sum_{j=k}^{n} (\Delta a_j) \cos kx$$

In this paper , we introduce new modified cosine sum and will study the L^1 convergence of this modified sum under the class S and C as follows:

$$g_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta \left(\frac{a_j}{j^2}\right) k^2 \cos kx$$
$$= \frac{a0}{2} + \sum_{k=1}^{n}\left(\frac{a_k}{k^2} - \frac{a(k+1)}{(k+1)^2} + \frac{a_n}{n^2} - \frac{a_{n+1}}{(n+1)^2} \right) k^2 coskx$$
$$= \frac{a_0}{2} + \sum_{k=1}^{n} a_k Coskx - \frac{a_{n+1}}{(n+1)^2}\sum_{k=1}^{n} k^2 coskx$$
$$= S_n(x) - \frac{a_{n+1}}{(n+1)^2}\sum_{k=1}^{n} k^2 coskx$$
$$= S_n(x) - \frac{a_{n+1}}{(n+1)^2}\left(-D_n''(x)\right)$$
$$= S_n(x) + \frac{a_{n+1}}{(n+1)^2}\left(D_n''(x)\right)$$

Under L^1 convergence, we will prove the following Main results:

Main Result 1: If a_k belongs to the class S , then $\| g - g_n \| = o(1)$ as $n \to \infty$ iff $o(n^r) = 1$

Main Result 2: If a_k belongs to class C and $\frac{n^2}{(n+1)^2} a_{n+1} \log n = o(1)$ then $\|g - g_n\| = o(1)$ as $n \to \infty$ iff $|a_{n+1} \log n| = o(1)$

2. Lemmas

We require the following Lemmas for the proof of our result:

2.1 Lemma[5]

If $| a_k | \leq 1$, then

$$\int_0^{\pi} \left| \sum_{k=0}^{n} a_k D_k(x) \right| dx \leq C(n+1)$$

Where C is a positive absolute constant.

2.2 Lemma [2,20]

The results mentioned in this lemma are well known.

If Dn(x) and Dn⁻(x) are Drichlet and Conjugate Drichlet Kernels respectively and are defined by

$D_n(x) = \frac{Sin\left(n+\frac{1}{2}\right)x}{2\,Sinx/2}$, $\bar{D}_n = \frac{\cos x/2-\cos\left(n+\frac{1}{2}\right)x}{2\sin x/2}$

Then as per [12]

(i) $\| D_n^r(x)\| = \frac{4}{\pi}\,(n^r \log n) + O(n^r)$, r= 0,1,2,3,……, where $D_n^r(x)$ represent r-th derivative of the Drichlet kernel.

(ii) $\| \bar{D}_n^r(x)\| = O\,(n^r \log n)$, r=0,1,2,3,……

Again if $K_n(x)$ denotes Fezer Kernel defined by

$K_n(x) = \frac{1}{n+1}\sum_{j=0}^{n} D_j(x)$, then

(a) (i) $D_n(x) = (n + 1)D_n(x) - (n + 1)K_n(x)$

(ii) $D_n^{r+1}(x) = (n+1)\, D_n^r(x) - (n+1)\, K_n^r(x)$

(b) (i) $\|K_n(x)\| = O(1)$ (ii) $\|K_n^r(x)\| = O(n^r)$

Proof of Main Result 1.

$|g(x) - g_n(x)| = S_n(x) + \frac{a_{n+1}}{(n+1)^2} D_n''(x)$

Applying Abel's Transform

$g(x)-g_n(x) = \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) - a_{n+1}D_n(x) + \frac{a_{n+1}}{(n+1)^2} D_n''(x)$

$= \sum_{k=n+1}^{\infty} a_k D_k(x) - a_{n+1}D_n(x) + \frac{a_{n+1}}{n+1} K_n'(x)$, by using Lemma (2.2)

Now, Making use of Abel's Transformation and Lemma (2.1), we have

$\int_0^{\pi} |g(x) - g_n(x)|dx \le \int_0^{\pi} |\sum_{k=n+1}^{\infty} \Delta a_k D_k(x)|\,dx - |a_{n+1}|\int_0^{\pi} |K_n(x)|\,dx - |\frac{a_{n+1}}{n+1}|\int_0^{\pi} |K_n'(x)|\,dx$

$= \int_0^{\pi} |\sum_{k=n+1}^{\infty} A_k \frac{\Delta a_k}{A_k} D_k(x)|\,dx - |a_{n+1}|\int_0^{\pi} |K_n(x)|\,dx - |\frac{a_{n+1}}{n+1}|\int_0^{\pi} |K_n'(x)|\,dx \le \int_0^{\pi} |\sum_{k=n+1}^{\infty} \Delta A_k \sum_{j=0}^{k} \frac{\Delta a_j}{A_j} D_k(x)|\,dx -$

$|a_{n+1}|\int_0^{\pi} |K_n(x)|\,dx |\frac{a_{n+1}}{n+1}|\int_0^{\pi} |K_n'(x)|\,dx$

$\le C\sum_{k=n+1}^{\infty}(k+1)\Delta A_k - |a_{n+1}|\int_0^{\pi} |K_n(x)|\,dx - |\frac{a_{n+1}}{n+1}|\int_0^{\pi} |K_n'(x)|\,dx$

The first term converges as per hypothesis, For 2nd term $\|K_n(x)\| = O(1)$,

For 3rd term $\|K_n^r(x)\| = O(n^r)$, r=0,1,2,3….

So, 3rd term converges iff $O(n^r) = 1$

Proof of Main Result 2.

$|g(x) - g_n(x)| = S_n(x) + \frac{a_{n+1}}{(n+1)^2} D_n''(x)$

Applying Abel's Transform

$g(x)-g_n(x) = \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) - a_{n+1}D_n(x) + \frac{a_{n+1}}{(n+1)^2} D_n''(x)$

Now, Making use of Abel's Transformation and Lemma (2.1), we have

$\int_0^\pi |g(x) - g_n(x)|dx \le \int_0^\pi |\sum_{k=n+1}^{\infty} \Delta a_k D_k(x)|\, dx + |a_{n+1}| \int_0^\pi |D_n(x)|dx + |\frac{a_{n+1}}{n+1}| \int_0^\pi |D_n^{''}(x)|\, dx$

$\| g(x) - g_n(x)\| \le \int_0^\pi |\sum_{k=n+1}^{\infty} \Delta a_k D_k(x)|\, dx + |a_{n+1}|\, \|D_n(x)\| + |\frac{a_{n+1}}{n+1}|\, \|D_n^{''}(x)\|$

The first term converges as per hypothesis acc. to definition 2. and for 2nd and 3rd term we will use Lemma 2.2(ii) and according to given condition $\| g- g_n\| = O(1)$.

We can also do extention of this modified sum as

$$g_n(x) = \frac{a0}{2} + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta \left(\frac{aj}{j^4}\right) k^4 \cos kx$$

$$= \frac{a0}{2} + \sum_{k=1}^{n} \left(\frac{a_k}{k^4} - \frac{a(k+1)}{(k+1)^4} + \frac{a_n}{n^4} - \frac{a_{n+1}}{(n+1)^4} \right) k^4 coskx$$

$$= \frac{a_0}{2} + \sum_{k=1}^{n} a_k Coskx - \frac{a_{n+1}}{(n+1)^4} \sum_{k=1}^{n} k^4 coskx$$

$$= S_n(x) - \frac{a_{n+1}}{(n+1)^4} \sum_{k=1}^{n} k^4 Coskx$$

$$= S_n(x) - \frac{a_{n+1}}{(n+1)^4} \left(D_n^{iv}(x)\right)$$

$$= S_n(x) - \frac{a_{n+1}}{(n+1)^4} \left(D_n^{iv}(x)\right)$$

Using Above modified sum, we will prove the following result:

Main Result 3. If a_k belongs to the class S and $|a_n \log n| = O(1)$ then $\| g- g_n\| = O(1)$ as $n \to \infty$ iff $O(n^r) = 1$

Proof of Main Result 3.

$| g(x) - g_n(x)| = S_n(x) - \frac{a_{n+1}}{(n+1)^4} D_n^{iv}(x)$

Applying Abel's transform

$g(x)-g_n(x) = \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) - a_{n+1}D_n(x) - \frac{a_{n+1}}{(n+1)^4} D_n^{iv}(x)$

$= \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) - 2a_{n+1}D_n(x) + a_{n+1}K_n(x) + \frac{a_{n+1}}{n+1} K_n^{'}(x) + \frac{a_{n+1}}{(n+1)^2} K_n^{''}(x) + \frac{a_{n+1}}{(n+1)^3} K_n^{'''}(x)$ By using Lemma (2.2)

Now, making use of Abel's Transformation and lemma (2.1), we have

$\int_0^\pi |g(x) - g_n(x)|dx \le \int_0^\pi |\sum_{k=n+1}^{\infty} \Delta a_k D_k(x)|\, dx + 2|a_{n+1}| \int_0^\pi |D_n(x)|dx + |a_{n+1}| \int_0^\pi |K_n(x)|\, dx + |\frac{a_{n+1}}{n+1}| \int_0^\pi |K_n^{'}(x)|dx + \left|\frac{a_{n+1}}{(n+1)^2}\right| \int_0^\pi |K_n^{''}(x)|dx + \left|\frac{a_{n+1}}{(n+1)^3}\right| \int_0^\pi |K_n^{'''}(x)|\, dx$

$= \int_0^\pi |\sum_{k=n+1}^{\infty} A_k \frac{\Delta a_k}{A_k} D_k(x)|\, dx + 2|a_{n+1}| \int_0^\pi |D_n(x)|dx + |a_{n+1}| \int_0^\pi |K_n(x)|\, dx + |\frac{a_{n+1}}{n+1}| \int_0^\pi |K_n^{'}(x)|dx + \left|\frac{a_{n+1}}{(n+1)^2}\right| \int_0^\pi |K_n^{''}(x)|dx + \left|\frac{a_{n+1}}{(n+1)^3}\right| \int_0^\pi |K_n^{'''}(x)|\, dx$

$\leq \int_0^{\pi} |\sum_{k=n+1}^{\infty} \Delta A_k \sum_{j=0}^{k} \frac{\Delta a_j}{A_j} D_j(x)|\, dx$ + 2|a_{n+1}| $\int_0^{\pi}|D_n(x)|dx$+ |a_{n+1}| $\int_0^{\pi} |K_n(x)|\, dx$ + $|\frac{a_{n+1}}{n+1}| \int_0^{\pi}|K_n^{'}(x)|dx + \left|\frac{a_{n+1}}{(n+1)^2}\right| \int_0^{\pi}|K_n^{''}(x)|dx + \left|\frac{a_{n+1}}{(n+1)^3}\right| \int_0^{\pi} |K_n^{'''}(x)|\, dx$

$\leq$ C $\sum_{k=n+1}^{\infty}(k+1)\Delta A_k$+ 2|$a_{n+1}$| $\int_0^{\pi}|D_n(x)|dx$+ |a_{n+1}| $\int_0^{\pi} |K_n(x)|\, dx$ + $|\frac{a_{n+1}}{n+1}| \int_0^{\pi}|K_n^{'}(x)|dx$ + $\left|\frac{a_{n+1}}{(n+1)^2}\right| \int_0^{\pi}|K_n^{''}(x)|dx + \left|\frac{a_{n+1}}{(n+1)^3}\right| \int_0^{\pi} |K_n^{'''}(x)|\, dx$

The first term converges as per hypothesis, 2nd term converges as per given condition, for 3rd term || $K_n(x)$|| = O(1) , for rest of the terms || $K_n^r(x)$|| = O (n^r) , r= 0,1,2,3,…….
So, the proof will be completed iff O(n^r) = 1.

We can also make generalization form of above modified sum as

$g_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta\left(\frac{a_j}{j^r}\right) k^r \cos kx$, r=1,2,3,4,…….

References:

[1] Bala R. and Ram B.,Trignometric series with semi-convex coefficients, Tamkang J. Math. Vol. 18, No. 1, (1987) pp. 75-84
[2]Bary N.K., A treatise on trigonometric series,Vol.I and Vol.II, Pergamon Press, London 1964.
[3]Braha N. L., Integrability and L1-convergence of certain cosine sums with third quasi hyper convex coefficients, Hacettepe J. of Mathematics and statistics, Vol. 42(6)(2013), 653-658.
[4]Fomin G.A., A class of trigonometric series, Math. Zametki 23 (1978), 213-222.
[5] Fomin G.A., On linear methods for summing Fourier series, Math. Sbornik, 66(1964), 144-152.
[6] Garrett J.W. and Stanojevi• c • C.V., On integrability and L1-convergence of certain cosine sums, Notices, Amer. Math. Soc., 22 (1975), A-166.
[7] Karanvir Singh, Kanak Modi , On L1 convergence of Modified Trignometric sums under some classes of coefficients (2017), 1965-1974.
[8] Kaur J. and Bhatia S. S., Convergence of new modi_ed trigonometric sums in the metric space L, The Journal of Non Linear Sciences and Applications, 1(3)(2008),179-188.
[9] Kaur K., Bhatia S. S. and Ram B., On L1-Convergence of certain Trigonometric Sums, Georgian journal of Mathematics, 1(11)(2004), 98-104.
[10] Kolmogorov A.N.,Sur I'ordere de grandeur des coe_cients de la series 7 de Fourier-Lebesque, Bull. Polon.Sci.Math.Astronom.Phys., (1923), 83-86.
[11] Krasniqi X.Z., On L1-convergence of sine and cosine modi_ed sums, Journal of Numerical Mathematics and Stochastics, 7(1) (2015), 94-102.
[12] Kumari S. and Ram B., L1-convergence of a modi_ed cosine sum, Indian Journal of Pure and Applied Math., 19 (1988), 1101-1104.
[13] Ram B., Convergence of certain cosine sums in the metric space L, Proc. Amer. Math. Soc., 66 (2) (1977), 258-260.
[14] Rees C.S. and Stanojevi• c • C.V., Necessary and Su_cient condition for the integrability of certain cosine sums, J. Math. Anal. Appl., 43 (1973), 579-586.
[15] Sandeep Kaur Chouhan, Jatinderdeep Kaur, S.S.Bhatia , L1 convergence of Modified Trignometric Sums ; 6 (2016) , 326-329

[16] Sheng S., The extension of theorems of • C.V. Stanojevi• c and V.B Stanojevi• c , Proc. Amer. Math. Soc., 110 (1990), 895-904.
[17] Sidon S., Hinreichende Bedingungen f¨ur den Fourier-Charakter einer Trigonometrischen Reihe, J. London Math. Soc., 14 (1939), 158-160.
[18] Telyakovskii S.A., On a su_cient condition of Sidon for the integrability of trigonometric series, Math. Zametki ,14 (1973), 317-328.
[19] Young W.H., On the Fourier series of bounded functions, Proc. London Math. Soc., 12(2) (1913), 41-70.
[20] Zygmund A., Trigonometric series, Vol. I, Vol. II, Univ. Press of Cambridge (1959).