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Queuing System with Feedback and Reneging Due to Impatient Customers and Urgent Call

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Abstract: This paper deals with queuing model having M-service channels in series in which feedback is permitted from each server of the model to all its previous service channels including the same server also. Reneging characteristics due to impatient behaviour of the customer and urgent call have been incorporated in the queue-model. Input process and service rates follow Poisson and exponential probability distributions respectively where the queue discipline is SIRO. After the description of the queue-model, we write its differential equations in transient form, reduce these equations independent of time and solve the reduced equations either by iterative method or by mathematical induction. Mean queue lengths are derived for infinite capacity when queue discipline is FIFO. This model finds its applications in administrative setups.

Keywords: Feedback, reneging, steady-state behaviour, Poisson law, exponential distribution, impatient customers, urgent call, mean-queue length.

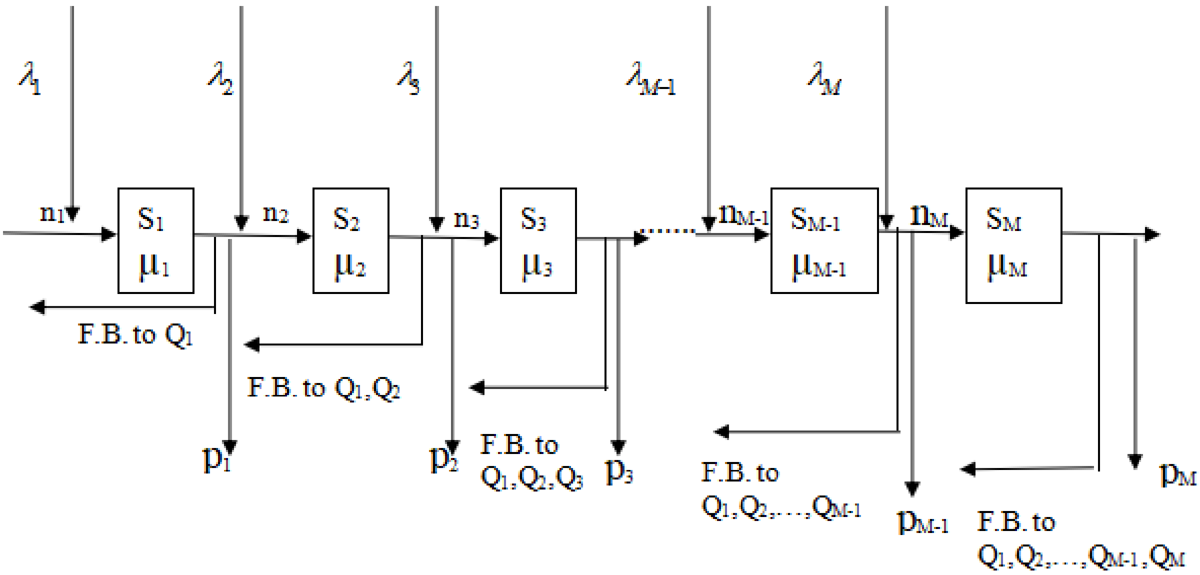
1. Introduction

O'Brien (1954), Jackson (1954) and Hunt (1955) analyzed the steady-state behaviour of serial queues with Poisson probability distribution. It has been assumed in these studies that the customers would join the system at the initial stage and pass through each service channels for the completion of service and are not allowed to renege or seek re-service. Barrer (1955) introduced reneging in the study of single service channel queuing model. Finch (1959) studied cyclic queues introducing feedback characteristic. Singh (1984) worked out on the steady-state behaviour of inpatient customers in the serial queuing model. Singh and Singh (2012) studied the problems of serial queues with reneging and feedback permissible from each server of the system to its previous service channel. Satyabirsingh (2016) analysed serial and non-serial queuing processes with various types of customers' behaviour. Sunder Rajan B et.al. (2017) worked on feedback queue with services in different stations under reneging and vacation policies. Meenu et.al. (2018) studied the customers' behaviour in multi-channel finite queuing system. Recently, Meenu, Singh and Deepak Gupta (2020) studied the customers' behaviour in multi-channel finite queuing system where feedback is permitted from the last channel to the last but one multiple channels only.Meenu, Singh and Deepak Gupta (2020) also did the time independent analysis of finite waiting space in multi-channel mixed queuing system with balking and reneging.The queuing model discussed by Singh and Singh (2012) has been modified by assuming that feedback is permissible from each server of the system to all its previous channels including the service channel of the server itself i.e. the same server.

The parameters which govern the present queuing model are given below:

- M-service channels are arranged in series and the customer may join each queue from outside directly and may leave the system at any stage.
- The customers may renege any queue of the system either due to their impatient behaviour or due to urgent call.
- Feedback is permissible to each server of the system to all its previous service channels including the queue before the same server.
- The input process follows Poisson law and service time distribution is exponential.
- The queue discipline is service in random order.
- The waiting space has infinite capacity.

1. Description of the Model:



Here, the queuing model is comprised of $Q_j\left(j=1,2,3,....,M\right)$ service channels with respective servers $S_j\left(j=1,2,3,....,M\right)$. The customers arrive $Q_j\left(j=1,2,3,....,M\right)$ from outside directly in Poisson stream with parameters $\lambda_j\left(j=1,2,......,M\right)$ and the service time distributions at the server S_j are distributed exponentially with parameter $\mu_j\left(j=1,2,3,....,M\right)$. It happens many times that the customer becomes impatient in the queue either due to long waiting time or large number of customers ahead of him or slow service rate of the server and then after a wait of certain time he may leave the queue without getting service or it happens also that the customer receives urgent call while waiting in the queue and leaves it without service. The reneging rates of the customers due to impatience after a wait of certain time T_{0j} and urgent call in the *jth* service channel are taken C_{jn_j} and r_j respectively where $C_{jn_j} = \frac{\mu_j e^{-\frac{\mu_j T_{0j}}{n_j}}}{1-e^{-\frac{\mu_j T_{0j}}{n_j}}} \left(j = 1,2,3,...,M \right)$. Here μ_j is the service rate and n_j is the queue size of $Q_j \left(j = 1,2,3,....,M \right)$. Further, after the completion of service at *jth* service channel, the customer either leaves the system with probability p_j or joins the next service channel with probability q_j or joins back all the previous channels including *jth* channel itself for re-service with probability $b_{ji}(i = 1,2,3,..., j)$ such that

$$p_j + q_j + \sum_{i=1}^{j} b_{ji} = 1(j = 1,2,3,...,M) \qquad (1)$$

2. Application:

The applications of such models are of common occurrence in various administrative departments. We consider here the hierarchy administrative set-up of a particular district(in a particular state) at the district head quarter consisting of Block development officer, Tehsildar, Sub-divisional magistrate, District magistrate etc. which corresponds to servers S_1, S_2, S_3, S_4 etc. of our model. The people meet the officers of the district regarding their problems from bottom to top. It has been observed that officers call the customers (people) for hearing randomly. The people may renege any office of the officer either due to heavy rush already there or urgent call. Senior officers may direct any concerned customer to meet his juniors from top to bottom if some informationregarding his problem are lacking. It may be possible that some officer may re-consider his case on his repeated request.

Marginal probabilities and mean queue lengths have been calculated for infinite waiting space under FIFO queue discipline.

3. Formulation of Equations: The probability $P\left(\tilde{n};t\right)$ is defined that at time 't' there are $n_1, n_2, n_3,..., n_M$ customers waiting (may renege due to impatience or due to urgent call) before $S_j \left(j = 1,2,3,....,M-1, M \right)$respectively.

We define the operators $T_j.,\ T._j$ *and* $T._{j,j+1}.$ on the vector $\tilde{n} = \left(n_1, n_2, n_3,.....n_M\right)$ in order to write the equations of the queuing model in the compact form

$$T_j.\left(\tilde{n}\right) = \left(n_1, n_2, n_3,....., n_j - 1,....n_M\right)$$

$$T._j\left(\tilde{n}\right) = \left(n_1, n_2, n_3,....., n_j + 1,...., n_M\right)$$

$$T._{j,j+1}.(\tilde{n}) = (n_1, n_2, n_3, \ldots.., n_j + 1, n_{j+1} - 1, \ldots., n_M)$$

4.1. Differential-difference equations:

Probability reasoning leads to following differential-difference equations

$$\frac{dP(\tilde{n};t)}{dt} = -\left[\sum_{j=1}^{M}\lambda_j + \sum_{j=1}^{M}\delta(n_j)\left(\mu_j + C_{jn_j} + r_j\right)\right]P(\tilde{n};t)$$

$$+\sum_{j=1}^{M}\lambda_j P(T_j.(\tilde{n});t) + \sum_{j=1}^{M-1}\mu_j q_j P(T._{j,j+1}.(\tilde{n});t) + \sum_{j=1}^{M}\left(\mu_j p_j + C_{jn_j+1} + r_j\right)P(T._j(\tilde{n});t)$$
$$+\sum_{j=1}^{M}\delta(m_j)(\mu_j b_{jj})P(\tilde{n};t) + \sum_{j=2}^{M}\mu_j\left(\sum_{i=1}^{j-1}b_{ji}P(n_1, n_2, \ldots, n_{i-1}, n_i - 1, n_{i+1}, \ldots, n_j + 1, \ldots, n_M; t)\right)$$

(2)

for $n_i \geq 0 \ (i = 1, 2, 3, \ldots., M)$

where $\delta(n_j) = \begin{cases} 1 & if \quad n_j \neq 0 \\ 0 & if \quad n_j = 0 \end{cases}$

and $P(\tilde{n};t) = \tilde{0}$ if any of the arguments is negative.

4.2. Steady-State Equations: The equations independent of time are written by taking the time derivatives to zero in the equation (2) as under

$$\left[\sum_{j=1}^{M}\lambda_j + \sum_{j=1}^{M}\delta(n_j)\left(\mu_j + C_{jn_j} + r_j\right)\right]P(\tilde{n}) = \sum_{j=1}^{M}\lambda_j P(T_j.(\tilde{n})) + \sum_{j=1}^{M-1}\mu_j q_j P(T._{j,j+1}.(\tilde{n}))$$
$$+\sum_{j=1}^{M}\left(\mu_j p_j + C_{jn_j+1} + r_j\right)P(T._j(\tilde{n})) + \sum_{j=1}^{M}\delta(m_j)(\mu_j b_{jj})P(\tilde{n})$$
$$+\sum_{j=2}^{M}\mu_j\left(\sum_{i=1}^{j-1}b_{ji}P(n_1, n_2, \ldots, n_{i-1}, n_i - 1, n_{i+1}, \ldots, n_j + 1, \ldots, n_M)\right)$$

For $n_j \geq 0 \ (j = 1, 2, 3, \ldots., M)$ (3)

Re-writing equation (3) as

$$\left[\sum_{j=1}^{M}\lambda_j+\sum_{j=1}^{M}\delta\left(n_j\right)\left(\mu_j\left(1-b_{jj}\right)+C_{jn_j}+r_j\right)\right]P\left(\tilde{n}\right)=\sum_{j=1}^{M}\lambda_j P\left(T_{j\cdot}\left(\tilde{n}\right)\right)+\sum_{j=1}^{M-1}\mu_j q_j P\left(T_{\cdot j,j+1\cdot}\left(\tilde{n}\right)\right)$$
$$+\sum_{j=1}^{M}\left(\mu_j p_j+C_{jn_j+1}+r_j\right)P\left(T_{\cdot j}\left(\tilde{n}\right)\right)$$
$$+\sum_{j=2}^{M}\mu_j\left(\sum_{i=1}^{j-1}b_{ji}P\left(n_1,n_2,\ldots,n_{i-1},n_i-1,n_{i+1},\ldots,n_j+1,\ldots,n_M\right)\right)$$

for $n_j\geq 0\ \left(j=1,2,3,\ldots.,M\right)$ (4)

4.3. Steady-State Solutions: The system of the Steady-State equations are satisfied by

$$P\left(\tilde{n}\right)=P\left(\tilde{0}\right)\left(\frac{\left(\lambda_1+\sum_{j=2}^{M}k_{j1}\rho_j\right)^{n_1}}{\prod_{j=1}^{n_1}\left(\mu_1\left(1-b_{11}\right)+C_{1j}+r_1\right)}\right)\cdot\left(\frac{\left(\lambda_2+\frac{\mu_1 q_1\rho_1}{h_1}+\sum_{j=3}^{M}k_{j2}\rho_j\right)^{n_2}}{\prod_{j=1}^{n_2}\left(\mu_2\left(1-b_{22}\right)+C_{2j}+r_2\right)}\right).$$

$$\left(\frac{\left(\lambda_3+\frac{\mu_2 q_2\rho_2}{h_2}+\sum_{j=4}^{M}k_{j3}\rho_j\right)^{n_3}}{\prod_{j=1}^{n_3}\left(\mu_3\left(1-b_{33}\right)+C_{3j}+r_3\right)}\right)\ldots.\left(\frac{\left(\lambda_{M-1}+\frac{\mu_{M-2}q_{M-2}\rho_{M-2}}{h_{M-2}}+k_{M,M-1}\rho_M\right)^{n_{M-1}}}{\prod_{j=1}^{n_{M-1}}\left(\mu_{M-1}\left(1-b_{M-1,M-1}\right)+C_{M-1,j}+r_{M-1}\right)}\right).$$
$$\left(\frac{\left(\lambda_M+\frac{\mu_{M-1}q_{M-1}\rho_{M-1}}{h_{M-1}}\right)^{n_M}}{\prod_{j=1}^{n_M}\left(\mu_M\left(1-b_{M,M}\right)+C_{M,j}+r_M\right)}\right).\quad \text{for } n_j\geq 0\ \left(j=1,2,3,\ldots.,M\right) \qquad (5)$$

Where

$$k_{ji}=\frac{b_{ji}\mu_j}{\mu_j\left(1-b_{jj}\right)+C_{jn_j+1}+r_j};\quad h_j=\mu_j\left(1-b_{jj}\right)+C_{jn_j+1}+r_j$$
$$\left(i=1,2,3,\ldots.,M-1;\ j=1,2,3,\ldots.,M\right) \qquad (6)$$

$\rho_1,\rho_2,\rho_3,\ldots,\rho_{M-1},\rho_M$ are unknown variables and are related to each other in the above result by the following relation

$$\rho_1=\lambda_1+k_{21}\rho_2+k_{31}\rho_3+k_{41}\rho_4+\ldots+k_{M-2,1}\rho_{M-2}+k_{M-1,1}\rho_{M-1}+k_{M,1}\rho_M$$

$$\rho_2 = \lambda_2 + \frac{\mu_1 q_1 \rho_1}{h_1} + k_{32}\rho_3 + k_{42}\rho_4 + ... + k_{M-2,2}\rho_{M-2} + k_{M-1,2}\rho_{M-1} + k_{M,2}\rho_M$$

$$\rho_3 = \lambda_3 + \frac{\mu_2 q_2 \rho_2}{h_2} + k_{43}\rho_4 + k_{53}\rho_5 + k_{63}\rho_6 ... + k_{M-2,3}\rho_{M-2} + k_{M-1,3}\rho_{M-1} + k_{M,3}\rho_M$$

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$$\rho_{M-2} = \lambda_{M-2} + \frac{\mu_{M-3} q_{M-3} \rho_{M-3}}{h_{M-3}} + k_{M-1,M-2}\rho_{M-1} + k_{M,M-2}\rho_M$$

$$\rho_{M-1} = \lambda_{M-1} + \frac{\mu_{M-2} q_{M-2} \rho_{M-2}}{h_{M-2}} + k_{M,M-1}\rho_M$$

$$\rho_M = \lambda_M + \frac{\mu_{M-1} q_{M-1} \rho_{M-1}}{h_{M-1}} \qquad (7)$$

M-equationsSolving these (7) M-equations for ρ_M with the help of determinants, we get

$$\rho_M = \frac{\begin{pmatrix} \lambda_M \Delta_{M-1} + \left(\frac{q_{M-1}\mu_{M-1}}{h_{M-1}}\right)\lambda_{M-1}\Delta_{M-2} \\ + \left(\frac{q_{M-1}\mu_{M-1}}{h_{M-1}}\right)\left(\frac{q_{M-2}\mu_{M-2}}{h_{M-2}}\right)\lambda_{M-2}\Delta_{M-3} + ... \\ + \left(\frac{q_{M-1}\mu_{M-1}}{h_{M-1}}\right)\left(\frac{q_{M-2}\mu_{M-2}}{h_{M-2}}\right).....\left(\frac{q_3\mu_3}{h_3}\right)\left(\frac{q_2\mu_2}{h_2}\right)\lambda_2\Delta_1 \\ + \left(\frac{q_{M-1}\mu_{M-1}}{h_{M-1}}\right)\left(\frac{q_{M-2}\mu_{M-2}}{h_{M-2}}\right).....\left(\frac{q_3\mu_3}{h_3}\right)\left(\frac{q_2\mu_2}{h_2}\right)\left(\frac{q_1\mu_1}{h_1}\right)\lambda_1 \end{pmatrix}}{\begin{pmatrix} \Delta_{M-1} - \left(\frac{q_{M-1}\mu_{M-1}}{h_{M-1}}\right)k_{M,M-1}\Delta_{M-2} \\ - \left(\frac{q_{M-1}\mu_{M-1}}{h_{M-1}}\right)\left(\frac{q_{M-2}\mu_{M-2}}{h_{M-2}}\right)k_{M,M-2}\Delta_{M-3} - ... \\ - \left(\frac{q_{M-1}\mu_{M-1}}{h_{M-1}}\right)\left(\frac{q_{M-2}\mu_{M-2}}{h_{M-2}}\right).....\left(\frac{q_2\mu_2}{h_2}\right)k_{M,2}\Delta_1 \\ - \left(\frac{q_{M-1}\mu_{M-1}}{h_{M-1}}\right)\left(\frac{q_{M-2}\mu_{M-2}}{h_{M-2}}\right).....\left(\frac{q_2\mu_2}{h_2}\right)\left(\frac{q_1\mu_1}{h_1}\right)k_{M,1} \end{pmatrix}} \qquad (8)$$

Where $$\Delta_M = \begin{vmatrix} 1 & -k_{21} & -k_{31} & \ldots & -k_{M-2,1} & -k_{M-1,1} & -k_{M,1} \\ \frac{-q_1\mu_1}{h_1} & 1 & -k_{32} & \ldots & -k_{M-2,2} & -k_{M-1,2} & -k_{M,2} \\ 0 & \frac{-q_2\mu_2}{h_2} & 1 & \ldots & -k_{M-2,3} & -k_{M-1,3} & -k_{M,3} \\ 0 & 0 & \frac{-q_3\mu_3}{h_3} & \ldots & -k_{M-2,4} & -k_{M-1,4} & -k_{M,4} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & \ldots & 1 & -k_{M-1,M-2} & -k_{M,M-2} \\ 0 & 0 & 0 & \ldots & \frac{-q_{M-2}\mu_{M-2}}{h_{M-2}} & 1 & -k_{M,M-1} \\ 0 & 0 & 0 & \ldots & 0 & \frac{-q_{M-1}\mu_{M-1}}{h_{M-1}} & 1 \end{vmatrix} \quad (9)$$ Then,

$$\Delta_{M-1} = \begin{vmatrix} 1 & -k_{21} & -k_{31} & \ldots & -k_{M-3,1} & -k_{M-2,1} & -k_{M-1,1} \\ \frac{-q_1\mu_1}{h_1} & 1 & -k_{32} & \ldots & -k_{M-3,2} & -k_{M-2,2} & -k_{M-1,2} \\ 0 & \frac{-q_2\mu_2}{h_2} & 1 & \ldots & -k_{M-3,3} & -k_{M-2,3} & -k_{M-1,3} \\ 0 & 0 & \frac{-q_3\mu_3}{h_3} & \ldots & -k_{M-3,4} & -k_{M-2,4} & -k_{M-1,4} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & \ldots & 1 & -k_{M-2,M-3} & -k_{M-1,M-3} \\ 0 & 0 & 0 & \ldots & \frac{-q_{M-3}\mu_{M-3}}{h_{M-3}} & 1 & -k_{M-1,M-2} \\ 0 & 0 & 0 & \ldots & 0 & \frac{-q_{M-2}\mu_{M-2}}{h_{M-2}} & 1 \end{vmatrix}$$

Continuing in this way, we get

$$\Delta_3 = \begin{vmatrix} 1 & -k_{21} & -k_{31} \\ \frac{-q_1\mu_1}{h_1} & 1 & -k_{32} \\ 0 & \frac{-q_2\mu_2}{h_2} & 1 \end{vmatrix},\ \Delta_2 = \begin{vmatrix} 1 & -k_{21} \\ \frac{-q_1\mu_1}{h_1} & 1 \end{vmatrix},\ \Delta_1 = |1| = 1$$

Since ρ_M has been derived so we can calculate ρ_{M-1} by putting the value of ρ_M in the last equation of (7), ρ_{M-2} by substituting the values of ρ_{M-1} and ρ_M in the last but one equation of (7). Proceeding in this way, we shall obtain $\rho_{M-3}, \rho_{M-4}, ---, \rho_3, \rho_2,$ and ρ_1,. In the solution (5) all the parameters $\rho_j \left(j = 1,2,3,...,M \right)$ have been determined except $P\left(\tilde{0}\right)$ which can be obtained by normalization $\sum_{\tilde{n}=\tilde{0}}^{\infty} P\left(\tilde{n}\right) = 1$ and with the restriction that each service channel's utilization factor is less than unity. Further, we discuss the queuing model under the situation when the reneging rates of the systems are independent of queue size and service rates and the customers are served with FIFO queue discipline. Then $C_{jn_j} = C_M$ for all

j=1,2,3,...,M. Putting $C_{jn_j} = C_j$ for all *j=1,2,3,...,M* in equations (2), (3), (4) then solution (5) of the system reduces to

$$P\left(\tilde{n}\right) = P\left(\tilde{0}\right)\left(\left(\frac{\lambda_1 + \sum_{j=2}^{M} k_{j1}\rho_j}{\left(\mu_1\left(1-b_{11}\right) + C_1 + r_1\right)}\right)^{n_1}\right).\left(\left(\frac{\lambda_2 + \frac{\mu_1 q_1 \rho_1}{h_1} + \sum_{j=3}^{M} k_{j2}\rho_j}{\left(\mu_2\left(1-b_{22}\right) + C_2 + r_2\right)}\right)^{n_2}\right).$$

$$\left(\left(\frac{\lambda_3 + \frac{\mu_2 q_2 \rho_2}{h_2} + \sum_{j=4}^{M} k_{j3}\rho_j}{\left(\mu_3\left(1-b_{33}\right) + C_3 + r_3\right)}\right)^{n_3}\right)....\left(\left(\frac{\lambda_{M-1} + \frac{\mu_{M-2} q_{M-2} \rho_{M-2}}{h_{M-2}} + k_{M,M-1}\rho_M}{\left(\mu_{M-1}\left(1-b_{M-1,M-1}\right) + C_{M-1} + r_{M-1}\right)}\right)^{n_{M-1}}\right).$$

$$\left(\left(\frac{\lambda_M + \frac{\mu_{M-1} q_{M-1} \rho_{M-1}}{h_{M-1}}}{\left(\mu_M\left(1-b_{M,M}\right) + C_M + r_M\right)}\right)^{n_M}\right).\ \text{for } n_j \geq 0 \ \left(j = 1,2,3,....,M \right) \qquad (10)$$

Where

$$k_{ji}=\frac{b_{ji}\mu_j}{\mu_j\left(1-b_{jj}\right)+C_j+r_j};\quad h_j=\mu_j\left(1-b_{jj}\right)+C_j+r_j \tag{11}$$

$$\left(i=1,2,3,....,M-1;\ j=1,2,3,....,M\right)$$

and ρ_M can be deduced directly from result (8) by assigning these values to k_{ji} and h_j in results (8) and (9) and $\rho_{M-1},\rho_{M-2},---,\rho_3,\rho_2,\rho_1$ would be evaluated by the same procedure mentioned earlier for result (5),We calculate $P\left(\tilde{0}\right)$ with the help of result (10) and normalization condition $\sum_{\tilde{n}=\tilde{0}}^{\infty}P\left(\tilde{n}\right)=1$ and the utilization factor less than unity as under

$$1=P\left(\tilde{0}\right)\left(\frac{1}{1-\left(\dfrac{\lambda_1+\sum_{j=2}^{M}k_{j1}\rho_j}{\mu_1\left(1-b_{11}\right)+C_1+r_1}\right)}\right).\left(\frac{1}{1-\left(\dfrac{\lambda_2+\dfrac{\mu_1q_1\rho_1}{h_1}+\sum_{j=3}^{M}k_{j2}\rho_j}{\mu_2\left(1-b_{22}\right)+C_2+r_2}\right)}\right)$$

$$.\left(\frac{1}{1-\left(\dfrac{\lambda_3+\dfrac{\mu_2q_2\rho_2}{h_2}+\sum_{j=4}^{M}k_{j3}\rho_j}{\mu_3\left(1-b_{33}\right)+C_3+r_3}\right)}\right)....\left(\frac{1}{1-\left(\dfrac{\lambda_{M-1}+\dfrac{\mu_{M-2}q_{M-2}\rho_{M-2}}{h_{M-2}}+k_{M,M-1}\rho_M}{\mu_{M-1}\left(1-b_{M-1,M-1}\right)+C_{M-1}+r_{M-1}}\right)}\right)$$

$$.\left(\frac{1}{1-\left(\frac{\lambda_M+\frac{\mu_{M-1}q_{M-1}\rho_{M-1}}{h_{M-1}}}{\mu_M\left(1-b_{M,M}\right)+C_M+r_M}\right)}\right)$$

Thus $$P\left(\tilde{0}\right)=\left(1-\left(\frac{\lambda_1+\sum_{j=2}^{M}k_{j1}\rho_j}{\mu_1\left(1-b_{11}\right)+C_1+r_1}\right)\right).\left(1-\left(\frac{\lambda_2+\frac{\mu_1 q_1\rho_1}{h_1}+\sum_{j=3}^{M}k_{j2}\rho_j}{\mu_2\left(1-b_{22}\right)+C_2+r_2}\right)\right).$$

$$.\left(1-\left(\frac{\lambda_3+\frac{\mu_2 q_2\rho_2}{h_2}+\sum_{j=4}^{M}k_{j3}\rho_j}{\mu_3\left(1-b_{33}\right)+C_3+r_3}\right)\right).....\left(1-\left(\frac{\lambda_{M-1}+\frac{\mu_{M-2}q_{M-2}\rho_{M-2}}{h_{M-2}}+k_{M,M-1}\rho_M}{\mu_{M-1}\left(1-b_{M-1,M-1}\right)+C_{M-1}+r_{M-1}}\right)\right)$$

$$.\left(1-\left(\frac{\lambda_M+\frac{\mu_{M-1}q_{M-1}\rho_{M-1}}{h_{M-1}}}{\mu_M\left(1-b_{M,M}\right)+C_M+r_M}\right)\right) \qquad (12)$$

4.4 Steady-State Marginal Probabilities:

$P\left(n_1\right)$ =(The marginal probability of the service channel before S_1) $=\sum_{n_2,n_3,\ldots n_M=0}^{\infty}P\left(\tilde{n}\right)$

Using the result (10) and (12), we get

$$P\left(n_1\right)=\left(1-\left(\frac{\lambda_1+\sum_{j=2}^{M}k_{j1}\rho_j}{\mu_1\left(1-b_{11}\right)+C_1+r_1}\right)\right).\left(\frac{\lambda_1+\sum_{j=2}^{M}k_{j1}\rho_j}{\mu_1\left(1-b_{11}\right)+C_1+r_1}\right)^{n_1} \qquad (13)$$

Similarly

$$P\left(n_i\right)=\left(1-\left(\frac{\lambda_i+\frac{\mu_{i-1}q_{i-1}\rho_{i-1}}{h_{i-1}}+\sum_{j=i+1}^{M}k_{ji}\rho_j}{\mu_i\left(1-b_{ii}\right)+C_i+r_i}\right)\right).\left(\frac{\lambda_i+\frac{\mu_{i-1}q_{i-1}\rho_{i-1}}{h_{i-1}}+\sum_{j=i+1}^{M}k_{ji}\rho_j}{\mu_i\left(1-b_{ii}\right)+C_i+r_i}\right)^{n_i}$$

$$\left(i=2,3,4,\ldots,M-1,M\right)$$

Mean Queue Length:

L_1 =(The Marginal Mean queue length before the server S_1) $=\sum_{n_1=0}^{\infty}n_1P\left(n_1\right)$

$$L_1=\frac{\left(\lambda_1+\sum_{j=2}^{M}k_{j1}\rho_j\right)}{\mu_1\left(1-b_{11}\right)+C_1+r_1-\left(\lambda_1+\sum_{j=2}^{M}k_{j1}\rho_j\right)}$$

Similarly for $i=2,3,4,\ldots,M-1,M$

$$L_i=\frac{\left(\lambda_i+\frac{\mu_{i-1}q_{i-1}\rho_{i-1}}{h_{i-1}}+\sum_{j=i+1}^{M}k_{ji}\rho_j\right)}{\left(\mu_i\left(1-b_{ii}\right)+C_i+r_i\right)-\left(\lambda_i+\frac{\mu_{i-1}q_{i-1}\rho_{i-1}}{h_{i-1}}+\sum_{j=i+1}^{M}k_{ji}\rho_j\right)}$$

Thus mean queue length is

$$L=\sum_{i=1}^{M}L_i$$

4. NUMERICAL ILLUSTRATION :

Taking the probability of joining the next server, arrival rate, service rate ,probability of joining the previous server , reneging rate and time taken to wait ; the mean queue length before the server is calculated.

Servers in Series S_M	Probability of Joining the Next Server q_M	Arrival rate λ_M before server S_M	Service rate μ_M before server S_M	Queue size	Probability of Joining the Previous Servers r_M including itself					Waiting time	Reneging rate due to impatience after a wait of time $T0_j$	Reneging rate due to urgent call			Marginal mean queue length before the server $S_M = L_M$
	q_M	λ_M	μ_M	n_M	b_{ji}					$T0_j$	C_M	r_i	Δ_M	ρ_M	L_M
					b1i	b2i	b3i	b4i	b5i						
1	0.01	1	35	5	0.13	0	0	0	0	500	0.003675	1	1	8.77771	0.387156
2	0.15	5	40	6	0.1	0.19	0	0	0	15	0.147662	3	0.99874	10.8416	0.441463
3	0.21	4	45	7	0.14	0.12	0.1	0	0	27	0.095575	5	0.97837	12.1791	0.36551
4	0.13	5	50	8	0.1	0.05	0.13	0.04	0	28	0.105323	4	0.95071	14.6226	0.391213
5	0.14	3	55	9	0.02	0.11	0.14	0.12	0.13	415	0.007976	7	0.93187	9.92641	0.220962
6	0.08	5	60	10	0.07	0.03	0.04	0.17	0.24	40	0.092061	3	0.8971	10.4969	0.2095062
7	0.05	6	65	11	0.15	0.14	0.13	0.23	0.06	45	0.090045	2	0.87803	9.51895	0.175526
8	0.17	5	70	12	0.03	0.01	0.02	0.07	0.15	50	0.08838	4	0.86566	6.24569	0.110437
9	0.25	7	75	13	0.08	0.07	0.12	0.13	0.09	13	0.369688	4	0.84896	8.1835	0.169373

Mean queue length of the system =2.47115

5.1 Algorithm for writing the program of given numerical of the model:

The following Algorithm provides the procedure to determine the mean queue length of the above given model.

STEP 1: Take the value of number of customers $(n_1, n_2, n_3, ...n_M)$

STEP 2: Take the probability of joining the next (i+1)th server $\left(q_1, q_2, q_3, ...q_M\right)$

STEP 3: Take the value of arrival rate $\left(\lambda_1, \lambda_2, \lambda_3, ...\lambda_M\right)$

STEP 4: Take the value of service rate ($\mu_1, \mu_2, \mu_3, ..., \mu_{M-1}, \mu_M$)

STEP 5: Take the value of probability of leaving the system ($\left(p_1, p_2, p_3, ...p_M\right)$

STEP 6: Take the value of probability of joining back all the previous channels including *jth* channel itself for re-service $b_{ji} (i = 1, 2, 3, ..., j)$

STEP 7: Take the value of waiting time $T0_j$ $\left(j = 1, 2, 3, ..., M\right)$

STEP8: Calculate the value of reneging rate :

$$C_{jn_j} = \frac{\mu_j e^{-\frac{\mu_j T_{0j}}{n_j}}}{1 - e^{-\frac{\mu_j T_{0j}}{n_j}}} \left(j = 1, 2, 3, ..., M\right)$$

STEP 9: Calculate the value of two unknowns.

$$k_{ji} = \frac{b_{ji}\mu_j}{\mu_j\left(1-b_{jj}\right)+C_{jn_j+1}+r_j}; \quad h_j = \mu_j\left(1-b_{jj}\right)+C_{jn_j+1}+r_j$$

STEP 10: Calculate the value of

$$\rho_1, \rho_2, \rho_3, \ldots, \rho_{M-1}, \rho_M$$

$$\rho_1 = \lambda_1 + k_{21}\rho_2 + k_{31}\rho_3 + k_{41}\rho_4 + \ldots + k_{M-2,1}\rho_{M-2} + k_{M-1,1}\rho_{M-1} + k_{M,1}\rho_M$$

$$\rho_2 = \lambda_2 + \frac{\mu_1 q_1 \rho_1}{h_1} + k_{32}\rho_3 + k_{42}\rho_4 + \ldots + k_{M-2,2}\rho_{M-2} + k_{M-1,2}\rho_{M-1} + k_{M,2}\rho_M$$

$$\rho_3 = \lambda_3 + \frac{\mu_2 q_2 \rho_2}{h_2} + k_{43}\rho_4 + k_{53}\rho_5 + k_{63}\rho_6 \ldots + k_{M-2,3}\rho_{M-2} + k_{M-1,3}\rho_{M-1} + k_{M,3}\rho_M$$

…………………………………

……………………………………….

$$\rho_{M-2} = \lambda_{M-2} + \frac{\mu_{M-3} q_{M-3} \rho_{M-3}}{h_{M-3}} + k_{M-1,M-2}\rho_{M-1} + k_{M,M-2}\rho_M$$

$$\rho_{M-1} = \lambda_{M-1} + \frac{\mu_{M-2} q_{M-2} \rho_{M-2}}{h_{M-2}} + k_{M,M-1}\rho_M$$

$$\rho_M = \lambda_M + \frac{\mu_{M-1} q_{M-1} \rho_{M-1}}{h_{M-1}} \, 1$$

STEP 11: Calculate the value of mean queue length

L_1 =(The Marginal Mean queue length before the server S_1) $= \sum_{n_1=0}^{\infty} n_1 P\left(n_1\right)$

$$L_1 = \frac{\left(\lambda_1 + \sum_{j=2}^{M} k_{j1}\rho_j\right)}{\mu_1\left(1-b_{11}\right) + C_1 + r_1 - \left(\lambda_1 + \sum_{j=2}^{M} k_{j1}\rho_j\right)}$$

Similarly for $i = 2, 3, 4, \ldots, M-1, M$

$$L_i = \frac{\left(\lambda_i + \frac{\mu_{i-1} q_{i-1} \rho_{i-1}}{h_{i-1}} + \sum_{j=i+1}^{M} k_{ji}\rho_j\right)}{\left(\mu_i\left(1-b_{ii}\right)+C_i+r_i\right)-\left(\lambda_i + \frac{\mu_{i-1} q_{i-1} \rho_{i-1}}{h_{i-1}} + \sum_{j=i+1}^{M} k_{ji}\rho_j\right)}$$

Thus mean queue length of the model is

$$L = \sum_{i=1}^{M} L_i$$

5.2 We can solve the above numerical by creating a Program for finding mean queue length of the model

```
M=9;
q[1]=0.01;□ [1]=1;□ [1]=35;n[1]=5;
q[2]=0.15;□ [2]=5;□ [2]=40;n[2]=6;
q[3]=0.21;□ [3]=4;□ [3]=45;n[3]=7;
q[4]=0.13;□ [4]=5;□ [4]=50;n[4]=8;
q[5]=0.14;□ [5]=3;□ [5]=55;n[5]=9;
q[6]=0.08;□ [6]=5;□ [6]=60;n[6]=10;
q[7]=0.05;□ [7]=6;□ [7]=65;n[7]=11;
q[8]=0.17;□ [8]=5;□ [8]=70;n[8]=12;
q[9]=0.25;□ [9]=7;□ [9]=75;n[9]=13;
r[1]=1;T0[1]=500;
r[2]=3;T0[2]=15;
r[3]=5;T0[3]=27;
r[4]=4;T0[4]=28;
r[5]=7;T0[5]=415;
r[6]=3;T0[6]=40;
r[7]=2;T0[7]=45;
r[8]=4;T0[8]=50;
r[9]=4;T0[9]=13;
```

```
b=({
   {0.13, 0, 0, 0, 0, 0, 0, 0, 0},
   {0.1, 0.19, 0, 0, 0, 0, 0, 0, 0},
   {0.14, 0.12, 0.1, 0, 0, 0, 0, 0, 0},
   {0.1, 0.05, 0.13, 0.04, 0, 0, 0, 0, 0},
   {0.02, 0.11, 0.14, 0.12, 0.13, 0, 0, 0, 0},
   {0.07, 0.03, 0.04, 0.17, 0.24, 0.04, 0, 0, 0},
   {0.15, 0.14, 0.13, 0.23, 0.06, 0.24, 0.05, 0, 0},
   {0.03, 0.01, 0.02, 0.07, 0.15, 0.13, 0.23, 0.16, 0},
   {0.08, 0.07, 0.12, 0.13, 0.09, 0.08, 0.1, 0.07, 0.3}
  });
For[i=1,i□ M,i++,For[j=1,j□ M,j++,g[i,j]=b[[i,j]]]]
(*Calculating C_j*)
For[j=1,j□ M,j++,c[j]=□ [j]*Exp[-□ [j]*T0[j]/(n[j])] /(1-Exp[-(□ [j]*T0[j])/(n[j])]) ]
(*Calculating k_{ji}*)
For[j=1,j□ M,j++,h[j]=□ [j] (1-g[j,j])+ c[j]+r[j] ]
For[j=1,j□ M,j++,For[i=1,i□ M,i++,k[j,i]=g[j,i]*□ [j] /(h[j])]]
For[i=1,i□ M,i++,
For[j=1,j□ M,j++,If[i□ j,f[i,j]=1
If[j>i, f[i,j]=-k[j,i],If[i□ j+1,f[i,j]=q[j]*□ [j]/h[j],f[i,j]=0]]]]]
(*Calculating □ _M*)
□ [0]=1;
For[t=1,t□ M,t++,□ [t]=Det[Table[f[i,j],{i,1,t},{j,1,t}]]];
(*Calculating ρ_M*)
ρ[M]=(λ[M]Δ[M−1]+Sum[(Product[q[i]∗μ[i]/h [i],{i,j,M−1}])Δ[j−1]λ[j],{j,M−1,1,−1}])/(Δ[M−1]−Sum[(Product[q[i]∗,μ[i]/h [i],{i,j,M−1}])Δ[j−1]∗k[M,j],{j,M−1,1,−1}]);
ρ[M−1]=(ρ[M]−λ[M]) (h [M−1]/(q[M−1]∗μ[M−1]));
For[t=M−2,t≥1,t+=−1,ρ[t]=(ρ[t+1]−λ[t+1]−Sum[(k[j,t+1]∗ρ[j]),{j,t+2,M}])(h [t]/(q[t]∗μ[t]))]
```

```
(*Calculating L_n*)
    l1[1]=,(λ[1]+Sum[(k[j,1]∗ρ[j]),{j,2,M}])/(h [1]−(λ[1]+Sum[(k[j,1]∗ρ[j]),{j,2,M}]));
    For[t=2,t≤M,t++,l1[t]=,(λ[t]+(q[t−1]∗μ[t−1]∗ρ[t−1]/h [t−1])+Sum[(k[j,t]∗ρ[j]),{j,t+1,M}])/(
  h [t]−(λ[t]+(q[t−1]∗μ[t−1]∗ρ[t−1]/h [t−1])+Sum[(k[j,t]∗ρ[j]),{j,t+1,M}]))]
For[t=1,t□ M,t++,Print[]]*)
   For[t=1,t□ M,t++,Print[ " □ _",t," is ", □ [t] , " □ _",t," is ", □ [t]," L_",t," is ", l1[t]]]
    (*Calculating L*)
     l=Sum[l1[t],{t,1,M}];
    Print["L is ",l]
```

Output:

□_1 is 1 □_1 is 8.77771 L_1 is 0.387156

□ □2 is 0.998743 □_2 is 10.8416 L_2 is 0.441463

□ □3 is 0.978366 □_3 is 12.1791 L_3 is 0.36551

□_4 is 0.950707 □_4 is 14.6226 L_4 is 0.391213

□_5 is 0.931865 □_5 is 9.92641 L_5 is 0.220962

□_6 is 0.897098 □_6 is 10.4969 L_6 is 0.209506

□_7 is 0.878025 □_7 is 9.51895 L_7 is 0.175526

□_8 is 0.865659 □_8 is 6.24569 L_8 is 0.110437

□_9 is 0.848958 □_9 is 8.1835 L_9 is 0.169373

Mean queue length of this model L = 2.47115

5. Concluding remarks:

1.The important concept of reneging has been introduced in the present study because reneging occurs either due to impatient behavior of the customer or due to urgent message and causes direct loss to business.

2. If feedback is limited from each service channel to its previous service channel in the present model, then result would resemble with the results of queuing model discussed by 19.

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