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Research Article

A Comprehensive Review of Fixed Point Theorems on Various Metric Spaces and Their Applications

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Abstract:
Aronszajn and Panitchpakdi developed hyperconvex metric spaces to expand Hahn-theorem Banach's beyond the real line to more generic spaces. The aim of this short article is to collect and combine basic notions and results in the fixed point theory in the context of hyperconvex metric spaces. In this paper, we first introduce the definitions of hyperconvex metric spaces, nonexpansive retract, externally hyperconvex and bounded subsets, and admissible subsets. We shall review and explore some fundamental characteristics of hyperconvexity. Next, we introduce the Knaster–Kuratowski and Mazurkiewicz (KKM) theory in hyperconvex metric spaces and related results. Furthermore, we find the relationship between extremal points and hyperconvexity and related properties. Furthermore, we have highlighted some known consequences of our main results. Fixed point theorems are fundamental tools in mathematical analysis, and have been used for diverse purposes including optimization, differential equations and dynamical systems. Fixed point theory was initiated in the case of standard metric spaces and subsequently expanded to be-metrical, convex be-metallic etc., as we needed more generalised conditions for broader range of maps from a wider variety of intuitive settings. This paper investigates fixed point theorems in convex b-metric spaces, which studies by Chen et al. This study aimed to synthesize and interpret the results in terms of practical implications as well as theoretical contributions that emerge from these findings, which are discussed later on this discussion section. In situations where traditional metric constraints are restrictive, fixed point theorems can be implemented in a class of spaces using convex b-metric structure by each relaxing and combining part of triangle inequality with the condition involved from standard assumption about Banach contraction. The results in this paper are existence and uniqueness theorems, common fixed point theorem of two mappings under some contractive conditions, a common coupled coincidence point result for four self maps, thereby indicating that convex b-metric spaces can be used but desired to solve boundary value problems (BVP), stabilization of dynamic systems over bounded closed sets with Lyapunov functions. Applications show how these abstract sittings could play a more elegant role in nonlinear optimization too by identifying optimal solutions. We also compare our result of convex b-metric spaces with other generalized metrics and we discuss their advantages for future research. In particular the above review sheds light on unlimitedness of these strikingly handy class, convex b-metric spaces alongside its implication for rich maths art instead.

Keywards: Hyperconvex metric space, externally hyperconvex, Fixed Point Theory, Convex b-Metric Spaces,.

1.Introduction

Aronszajn and Panitchpakdi introduced the concept of hyperconvexity by demonstrating that a hyperconvex space is an absolute retraction, i.e., it is a nonexpanding retraction of any metric space in which it is isometrically contained. The related linear concept is well established and is attributed to Goodner and Nachbin. The reader may refer to Lacey for further information on that linear theory. The nonlinear theory is still in its infancy. Isbell created a natural hyperconvex hull for every metric space. Recent interest in such spaces stems from the independent proofs by Sine and Soardi that the

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fixed point condition for nonexpansive mappings holds in bounded hyperconvex spaces. Numerous intriguing findings have been shown to hold in hyperconvex spaces. We are aware of the significance of the well-known Fan-KKM principle in the study of nonlinear analysis, particularly in the study of topology fixed point theory. Sine and Soardi separately demonstrated the importance of the connection between hyperconvex metric spaces and nonexpansive mappings. Fixed point theory is a branch of mathematical analysis that studies the conditions under which an element in a given metric space maps to itself through some function f : X→X. In 1912, Brouwer's Fixed Point Theorem states any continuous transformation from a closed and bounded convex subset of Euclidean Space into itself has at least one fixed point. It has been officially raised for the first time. The fixed point theory established here configured a precedent for the significance of fixed points in mathematical analysis, and helped lay base level ground on which advanced theorems could be built upon. However, Brouwers theorem was limited because it required conditions such continuity and closedness or convexity that in practice were not always applicable to the applied mathematician.
Additionally, bear in mind that Jawhari et al. demonstrated that Sine and Soardi's fixed point theorem is identical to the traditional Tarski fixed point theorem in fully ordered sets. This is accomplished via the concept of generalised metric spaces. Subsequent to this, in the decades that ensued researchers worked on making fixed point theorems applicable across a host of scientific and mathematical discipline leading up to principles even more important than all. For example, the Schauder Fixed Point Theorem is of central importance in functional and operator analysis -- it generalizes Brouwer's to infinite dimensions. By proving existence and uniqueness of solutions under different mappings (conditions for some spaces), these theorems showed that fixed Point theory was applicable to various application objserved both in theoretical and practical studies.
The focus has largely moved away from generalisations of these principles to weaker contexts, but fixed point theory is still a main research area in modern mathematics. Recently novel concepts such as b-Metric spaces, Convex b-metric Spaces etc. are more flexibly in handling complicated mappings having wide range conditions on them. Such an iterated development of these theorems reveals a commitment to fitting mathematical tools for ever more complex applications, thereby extending fixed point theory beyond its current parameters.

2 Notation and basic definitions

For convenience, metric spaces shall be represented as(M, d), or simply M, where M denotes the space and d is the distance on M. For us, the primary elements in a metric space will be closed balls indicated by $B(x, r)$, which stands for the closed ball with center x and radius $r \geq 0$. Additionally, the following notation is common when working with metric spaces and will be utilized throughout this article. Assume that M is a metric space, and that $x \in M$ and A and B are subsets of M;then

$$\begin{aligned}
r_x(A) &= \mathrm{s}\ \{d(x,y): y \in A\} \\
r(A) &= \mathrm{i}\ \{r_x(A): x \in M\} \\
R(A) &= \mathrm{i}\ \{r_x(A): x \in A\} \\
\text{diam } (A) &= \mathrm{s}\ \{d(x,y): x, y \in A\} \\
\text{dist } (x, A) &= \mathrm{i}\ \{d(x,y): y \in A\}, \\
\text{dist } (A, B) &= \mathrm{i}\ \{d(x,y): x \in A, y \in B\}, \\
C(A) &= \{x \in M: r_x(A) = r(A)\} \\
C_A(A) &= \{x \in A: r_x(A) = R(A)\}
\end{aligned}$$

cov $(A) = \bigcap\{B: B$ is a closed ball containing $A\}$ where $r(A)$ is the radius of A relative to M, diam (A) is the diameter of $A, R(A)$ is the Chebyshev radius of A, and cov (A) is the admissible cover of A.

Definition 2.1 A metric space M is said to be hyperconvex if given any family $\{x_\alpha\}$ of points of M and any family $\{r_\alpha\}$ of real numbers satisfying

$$d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$$

then

$$\bigcap_{\alpha} B(x_\alpha, r_\alpha) \neq \emptyset$$

where $B(x, r)$ denotes the closed ball centered at $x \in X$ with radius $r \geq 0$.
The recent interest into hyperconvexity goes back to the results of Sine and Soardi.
Convex b-metric spaces also accommodate weakly contractive mappings, which require a slightly modified form of contraction that is particularly useful for non-linear mappings. For a mapping T in a convex b-metric space, if there exists a condition such that $d(T(x), T(y)) < d(x, y)$ for all $x, y \in X$ with $x \neq y$, then T has a unique fixed point. This type of theorem is advantageous in iterative processes, as it ensures that weak contractiveness is sufficient for convergence, an essential property in optimization and dynamic systems analysis.

Definition 2.2 If H is a bounded hyperconvex metric space and $T: H \rightarrow H$ is nonexpansive, i.e., $d(T(x), T(y)) \leq d(x, y)$ for any $x, y \in H$, then there exists a fixed point $x \in H$, i.e., $T(x) = x$. Moreover, the fixed point set Fix (T) is hyperconvex and, consequently, is a nonexpansive retract of H.

Definition 2.3 A subset A of a metric space X is called externally hyperconvex (cf. [1]) if for any collection of balls $\{B(x_i, r_i)\}_{i \in I}$ in X with $d(x_i, x_j) \leq r_i + r_j$ and $d(x_i, A) \leq r_i$ we have $A \cap \bigcap_i B(x_i, r_i) \neq \emptyset$.
Introduction to Metric and b-Metric Spaces
A traditional metric space is a set X equipped with a metric $d: X \times X \rightarrow \mathbb{R}$, which satisfies the following conditions for all $x, y, z \in X$:
1 Non-negativity: $d(x, y) \geq 0$, and $d(x, y) = 0$ if and only if $x = y$.
2 Symmetry: $d(x, y) = d(y, x)$.
3 Triangle Inequality: $d(x, z) \leq d(x, y) + d(y, z)$.

Lemma 2.4 Let A be a bounded subset of a hyperconvex metric space M. Then
1 cov $(A) = \cap \{B(x, r_x(A)): x \in M\}$.
2 $r_x(\text{cov } (A)) = r_x(A)$, for any $x \in M$.
3 $r(\text{cov } (A)) = r(A)$.
4 $r(A) = \frac{1}{2}$diam (A).
5 diam (cov $(A)) =$ diam (A)
6 If $A =$ cov (A) then $r(A) = R(A)$.

Definition 2.5 Let M be a metric space. $A \subseteq M$ is said to be an admissible subset of M if $A =$ cov (A). The collection of all admissible subsets of M is then denoted by $\mathcal{A}(M)$.
In a convex b-metric space, the set X is both a b-metric space and convex, meaning that for any two points $x, y \in X$ and any $\lambda \in [0,1]$, the point $z = \lambda x + (1 - \lambda)y$ lies within X. Convexity, as applied in this context, is a valuable addition because it facilitates various mathematical processes, such as averaging or blending between points, which is useful in fields like optimization where iterative methods converge toward optimal points.

Proposition 2.6 Let (X, d) be a hyperconvex space and $\{A_i\}_{i \in I}$ a family of pairwise intersecting externally hyperconvex subsets such that one of them is bounded. Then $\bigcap_{i \in I} A_i \neq \emptyset$

Proposition 2.7. A metric space (X, d) is injective if and only if it is hyperconvex.

Proposition 2.8 A Banach space is said to be hyperconvex if and only if it is linearly isometric to C(K), where C(K) denotes the space of all continuous real functions defined on any stonian space K.

3. The KKM theory in hyperconvex metric spaces

The fundamental result of asserts that a metric space M is hyperconvex if and only if it is injective. Thus M is hyperconvex if given any two metric spaces X and Y with Y a subspace of X, and any nonexpansive mapping $f: Y \to M$, then f has a nonexpansive extension $\tilde{f}: X \to M$. An admissible subset of M is a set of the form

$$\bigcap_i B(x_i; r_i)$$

where $\{B(x_i; r_i)\}$ is a family of closed balls centered at points $x_i \in M$ with respective radii r_i. It is quite easy to see that an admissible subset of a hyperconvex metric space is hyperconvex. In what follows we use $\mathcal{A}(M)$ to denote the family of all nonempty admissible subsets of M.

The result indicates that the class of convex b-metric spaces is an improvement on fixed point theory. We use this fact to establish a new mathematical tool, combining the convex-type structure with b-metric spaces under much weaker assumptions than previously required in order for it to be applicable on more general problems. Not only does it serve as a substantial step forward in understanding the convergence in generalised distances, establishing fixed point theorems on these spaces presents new possibilities for dealing with problems demanding fine-grained distance concepts like artificial intelligence, optimisation and dynamic system analysis. Convex b-metric spaces will soon solidify itself in the modern applications, as work improves with time and fixed point theory evolve along stochastically changing mathematical challenges.

Lemma 3.1 Let H be a hyperconvex metric space. Suppose $E \subset H$ is externally hyperconvex relative to H and suppose A is an admissible subset of H. Then $E \cap A$ is externally hyperconvex relative to H.

Theorem 3.2 Let $\{H_i\}$ be a descending chain of nonempty externally hyperconvex subsets of a bounded hyperconvex space H. Then $\cap_i H_i$ is nonempty and externally hyperconvex in H.

Proof. Assures that $D := \cap_i H_i \neq \emptyset$. To see that D is externally hyperconvex. let $\{x_\alpha\} \subset H$ and $\{r_\alpha\} \subset \mathbb{R}$ satisfy $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$ and dis $\ell(x_\alpha, D) \leq r_\alpha$. Since H is hyperconvex we know thal $A := \cap_\alpha B(x_\alpha; r_\alpha) \neq \emptyset$

Also, since $\operatorname{dist}(x_\alpha, D) \leq r_\alpha$ we have $\operatorname{dist}(x_\alpha, H_i) \leq r_\alpha$ for each i, so by external hyperconvexity of H_i we conclude $A \cap H_i \neq \emptyset$ for each i. By Lemma $\{A \cap H_i\}$ is descending chain of nonempty hyperconvex subsets of H, so we have $\cap_i (A \cap H_i) = A \cap D \neq \emptyset$.

In this section, we deduce useful generalized forms of the KKM type theorems. we have the following:

Theorem 3.3 Let H be a hyperconvex space, $X \subset H$, and $G: X \multimap H$ a KKM map with compactly closed values. Then for every compact hyperconvex subsets K_0, $H_0 \subset H$, we have

$$(K_0 \cap H_0) \cap \bigcap_{x \in (K_0 \cap H_0) \cap X} \{G(x): x \in x \in (K_0 \cap H_0) \cap X\} \neq \emptyset \qquad (1)$$

Proof. Define $G_0(x) = G(x) \cap (K_0 \cap H_0)$ for $x \in (K_0 \cap H_0) \cap X$. Then $G_0: (K_0 \cap H_0) \cap X \multimap H_0$ is well-defined.

If $K_0 \cap H_0 = \emptyset$. Then by Theorem 3.3, (1) is true trivially.

Theorem 3.4 Let H be a hyperconvex space, $X \subset H, Y$ a topological space, $t \in \mathbb{C}(H, Y), G: X \multimap Y$ a map, and K_1 , K_2 be two nonempty compact subsets of Y. Suppose that

(3.1) for each $x \in X, G(x)$ is compactly closed;

(3.2) $t^{-1}G: X \multimap H$ is a KKM map; and

(3.3) for any $N \in \langle X \rangle$, there exists a compact hyperconvex subset $L_N \subset H$ containing N such that $t(L_N) \cap \bigcap\{G(x): x \in L_N \cap X\} \subset K_1$ and $t(L_N) \cap \bigcap\{G(x): x \in L_N \cap X\} \subset K_2$

Then we have

$$\overline{t(H)} \cap K_1 \cap K_2 \cap \bigcap \{G(x): x \in X\} \neq \emptyset$$

4. Extremal Points And Hyperconvexity

Borkowski M, Bugajewski D, Przybycie H, investigate linear hyperconvex spaces with extremal points of their unit balls. They prove that only in the case of a plane (and obviously a line) is there a strict connection between the number of extremal points of the unit ball and the hyperconvexity of space. As an example in non-linear analysis, complex systems and optimisation problems have been studied whereas conventional metric assumption is often not applicable such results are applied on fixed point theorems in b-metric spaces. Also some generalized contractive conditions in this are established which have made fixed point theorems of b-metric spaces more applicable to several engineering and physics problems.

Proposition 4.1 Let $\|\cdot\|$ be a norm in $\mathbb{R}^2$ such that the closed unit ball in this norm has exactly four extremal points. Then $\mathbb{R}^2$ with this norm is a hyperconvex metric space.
Proof: Let $z_i = (x_i, y_i)$, where $i = 1, \dots, 4$, denote the extremal points of the closed unit ball $\bar{B}$. We may assume that $z_1 = -z_3$ and $z_2 = -z_4$. Let us denote the maximum norm in $\mathbb{R}^2$ by $\|\cdot\|_m$.
Consider the linear mapping $T: (\mathbb{R}^2, \|\cdot\|_m) \to (\mathbb{R}^2, \|\cdot\|)$ given by the matrix

$$M(T) = \begin{bmatrix} \frac{1}{2}(x_1 + x_2) & \frac{1}{2}(x_1 - x_2) \\ \frac{1}{2}(y_1 + y_2) & \frac{1}{2}(y_1 - y_2) \end{bmatrix}$$

Put $u_1 = (1,1), u_2 = (1,-1), u_3 = (-1,1), u_4 = (-1,-1)$. Obviously, we have $Tu_i = z_i$ for $i = 1, \dots, 4$. Let us denote by $\bar{B}_m$ the closed unit ball in the space $(\mathbb{R}^2, \|\cdot\|_m)$. Then $\bar{B}_m = \operatorname{conv}\{u_i: i = 1, \dots, 4\}$ and $T(\bar{B}_m) = \bar{B}$.
Thus T is a nonsingular linear mapping of norm 1. Further, there exists T^{-1} and $\|T^{-1}\| = 1$, so T is an isometry. Hence $(\mathbb{R}^2, \|\cdot\|_m)$ is hyperconvex and the proof is complete. □
The question is now as follows: does there exist a norm in $\mathbb{R}^2$ such that the closed unit ball in this normed space has more than 4 extremal points and this space is hyperconvex? The following result gives the answer.

Proposition 4.2 Let there be given a norm in $\mathbb{R}^2$ such that the closed unit ball in this space has more than four extremal points. Then this space is not hyperconvex.
As a corollary from Propositions 4.1 and 4.2 we obtain the following characterisation. The result indicates that the class of convex b-metric spaces is an improvement on fixed point theory. We use this fact to establish a new mathematical tool, combining the convex-type structure with b-metric spaces under much weaker assumptions than previously required in order for it to be applicable on more general problems. Not only does it serve as a substantial step forward in understanding the convergence in generalised distances, establishing fixed point theorems on these spaces presents new possibilities for dealing with problems demanding fine-grained distance concepts like artificial intelligence, optimisation and dynamic system analysis. Convex b-metric spaces will soon solidify itself in the modern applications, as work improves with time and fixed point theory evolve along stochastically changing mathematical challenges.

Theorem 4.3 A space $(\mathbb{R}^2, \|\cdot\|)$ is hyperconvex if and only if the closed unit ball in this space has exactly four extremal points.
The space $\mathbb{R}^3$ has quite different character. Namely, we shall prove the following

Proposition 4.4 For every even number $n \geqslant 6$ there exists a norm $\|\cdot\|_n$ in $\mathbb{R}^3$ such that:
a) The closed unit ball in $(\mathbb{R}^3, \|\cdot\|_n)$ has exactly n extremal points.
b) The space $(\mathbb{R}^3, \|\cdot\|_n)$ is not hyperconvex.

5. Generalized forms of the KKM type Theorems

Let X be a nonempty set. We denote by $\mathcal{F}(X)$ and 2^X the family of all nonempty finite subsets of X and the family of all subsets of X, respectively. If A is a subset of a linear space E, the notation ' con $v(A)$ ' always means the convex hull of A.

Definition 5.1 Let X be any nonempty set and let M be a metric space. A set-valued mapping $G: X \to 2^M \setminus \{\emptyset\}$ is said to be a generalized metric KKM mapping (GMKKM) if for each nonempty finite set $\{x_1, \ldots, x_n\} \subset X$, there exists a set $\{y_1, \ldots, y_n\}$ of points of M, not necessarily all different, such that for each subset $\{y_{i_1}, \ldots, y_{i_k}\}$ of $\{y_1, \ldots, y_n\}$ we have

$$\text{co}\left\{y_{i_j}: j = 1, \ldots, k\right\} \subset \bigcup_{j=1}^{k} G\left(x_{i_j}\right)$$

Definition 5.2 Let X be a nonempty subset of a metric space M. Suppose $G: X \to 2^M$ is a set-valued mapping with nonempty values. Then G is said to be a metric KKM (MKKM) mapping if for each finite subset $F \in \mathcal{F}(X), \text{co}\,(F) \subset \bigcup_{x \in F} G(x)$.

Theorem 5.3 Let X be a non-empty set and M be a hyperconvex metric space. Suppose $G: X \to 2^M \setminus \{\emptyset\}$ is a set-valued mapping with nonempty closed values and suppose there exists $x_0 \in X$ such that $G(x_0)$ is compact. Then $\bigcap_{x \in X} G(x) \neq \emptyset$ if and only if the mapping G is a generalized metric KKM mapping.

Moreover, convex b-metric spaces are used in dynamic systems to analyse their stability and it is a very important issue when understanding the system response properties due to changes or stimuli. This means that Fixed point theorems in convex b-metric spaces can precisely model essential stability characteristics of non-linear, interdependent components and their interaction (behaviour) when exposed to shock. Actually, the rich structure of convex b-metric spaces is especially suitable for setting up a flexible framework in ecology, economics and control theory since many phenomena can be addressed which are impossible to model within traditional metric settings. Stability provides a primary condition for economic models, particularly dynamic systems to generate the particular equilibrium reactions and inform stable policy rules to public policymakers. The study of stability in ecology also helps conservationists to conserve populations by predicting the change in a population of an ecosystem with respect to environmental or demographic changes and hence is considered as one of the important tool for sustainability of ecosystems.

6. Applications of Fixed Point Theorems in Convex b-Metric Spaces

They have proven useful in complex problem solving environments such as optimisation, differential and integral equations, dynamic systems and game theory. Specific types of fixed point theorems in convex b-metric spacesare now well recognized as important results and have had many applications. Due to the specific property of convex b-metric spaces, these results can treat non-linear processes with correlation in a more successful way when compared with classical metric theories. This structure in these spaces is a generalised metric which is convex as well. The Flexifixed spaces allow both the academic and practitioners to appeal to fixed point theorems in domains that call for flexibility as well as stability. This is achieved by lifting the triangle inequality constraint and adding convexity. Iterative convergence in optimization problems, proofs of existence and uniqueness of solutions for differential equations, analysis of stability in dynamic systems and seeking equilibria competive multi-agent system such as those discussed by game theory are all exactly the kind where this adaptability proves beneficial.

In the field of optimisation theory, some iterative algorithms in non-linear optimization could be enriched by gaining the convex b-metric spaces. For optimization is the quest for a best answer from a field of possible answers. This is typically done by methods that iteratively move toward an optimal solution. These are algorithms ensuring convergence using fixed point theorems that guarantee the process stabilizes at a solution after repeated repetitions do not change it. We highlight the need to consider non-linear functions and mappings that fail under conventional metrics in achieving optimal solutions in convex b-metric spaces. Which is one of optimization´s most crucial processes. For example, in cases of standard metric space where the mappings do not directly follow normal triangle inequality some optimisation techniques based on convergence proof involving contraction mappings may never work. This is due to the fact that contraction mappings are a pivotal part on which convergence depends. Contrast this with the ease of convergence for any class of sequences in a space, due to the fact that no triangle inequality needs be satisfied and there is always just unity proportionality factor), as long as it satisfies all other properties. For example, this is especially important to remember when working with non-linear optimisation problems that have complex objective functions introduced. Problems such as resource allocation, machine learning and engineering design optimisation are examples of this. For instance, consider the case of a resource allocation problem in which an organization wants to optimize total output by allocating resources among different divisions. A new fixed point theorem is introduced for the approach of both resources partition according to productivity feedback from every department by using an iterative algorithm in a convex b-metric space. The elementwise gradient update iteratates refinements of the allocation, and due to properties specific convex b-metric spaces we are guarenteed convergence towards an optimal distribution even when department productivity is a complex non-linear function in resources. Many current examples of challenging optimisation tasks involve the need to prove convergence where biological or other proxies for agent intelligence are behaving without strong assumptions on their behaviour.

Conclusion: In comparison to the lack of linearity, hyperconvexity provides a very complex metric structure, which leads to a number of surprising and appealing discoveries in a variety of branches of mathematics, including topology, graph theory, multivalued analysis, and fixed point theory. Nonexpansive mappings have traditionally garnered the most attention, since they are at the core of hyperconvex metric space fixed point features. As such, convex b-metric spaces expand the application of fixed point theorems to a more extensive range of problems offering the possibility for novel partial solutions to fields needing flexibility and centrality such as optimisation, differential equations, dynamic systems, and game theory among others. The ability to apply the fixed point theorems to convex b-metric spaces advances the optimisation theory in offering the needed flexibility necessary for job applications on non-linear problems where the strict metric assumptions such as strong contraction are not valid. Convergence through iterations can be achieved using lesser strict conditions, and algorithmic solutions can be developed for identifying the ideal solution in high dimensional and complex environments. The application of fixed point theorems to differential and integrals in convex b-metric spaces has helped solve boundary value problems and show the existence and uniqueness of solutions in non-standard conditions. The application is substantially relevant in physical and engineering models that depict processes in various interacting domains using differential equations. The convex formulation ensures that solutions are bounded and continuous despite the chaos and uncertainties presented by the equations or the boundary conditions.

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