Turkish Online Journal of Qualitative Inquiry (TOJQI)
Volume 12, Issue 7, July 2021: 11285 – 11289

Research Article

Common Fixed Point Theorem For Four Compatible And Subsequentially Continuous Maps In G-Metric Spaces

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Abstract:
In this manuscript, our aim is to prove a new common fixed point theorem for four compatible and subsequentially continuous (alternately sub compatible and reciprocally continuous) maps in the G-metric spaces satisfying a more generalized contractive condition.

Keywords: *Common fixed point, compatibility, G-metric spaces.*

1. Introduction:

The notion of G-metric spaces was introduced by Mustafa and Sims [6]. After that a lot of authors have worked in this direction [see 7-10].

Following definitions will be used in sequel:

Definition 1.1[6] G-metric spaces:

In 2006, Mustafa and Sims introduced the concept of G-metric space as follows:

Let X be a nonempty set, and $let\ G : X \times X \times X \rightarrow \mathbb{R}^+$ be a function satisfying the following:

(G1) $G(x,y,z) = 0\ if\ x = y = z,$

(G2) $0 < G(x,x,y)\ for\ all\ x,y\ in\ X\ with\ x \neq y,$

(G3) $G(x,x,y) \leq G(x,y,z)\ for\ all\ x,y,z\ in\ X\ with\ z \neq y,$

$(G4)$ $(x,z,y) = (\ G(x,y,z) = G(y,z,x) = \ldots$ (symmetry in all three variables),

(G5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)\ for\ all\ x,y,z,a\ in\ X$ (rectangle inequality).

Then the function G is called a G-metric on X and the pair (X,G) is called a G-metric space.

Definition 1.2[6] If $G(x,y,y) = G(y,x,x)\ \forall\ x,y \in X$, then (X,G) is called a symmetric $G-$ metric space.

Definition 1.3[1] Let (X,G) be a G- metric space and S and T be two self maps on X. Then S and T are said to be compatible if

$\lim_{n\to\infty} G\big(STx_{n,}, TSx_n, TSx_n\big) = 0$, whenever $\{x_n\}$ is a sequence in X such that

$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$.

Definition 1.4[2] Two self mappings S and T are said to be conditionally reciprocally continuous, if whenever the set of sequences $\{x_n\}$ in X satisfying $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n$ is nonempty, there exit a sequence $\{y_n\}$ in X satisfying $\lim_{n\to\infty} S\, y_n = \lim_{n\to\infty} Ty_n = t$ (say) such that $\lim_{n\to\infty} STy_n = St \ and \ \lim_{n\to\infty} TSy_n = Tt$.

Definition 1.5[2] A pair of self mappings S and T is said to be reciprocally continuous if $\lim_{n\to\infty} STx_n = St$, $\lim_{n\to\infty} TSx_n = Tt$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$, for some $t \ in \ X$.

Definition 1.6[5] A pair of self mappings S and T is said to be subcompatible if $\lim_{n\to\infty} Sx_n = t$, $\lim_{n\to\infty} Tx_n = t$ whenever $\{x_n\}$ is a sequece in X and $\lim_{n\to\infty} G\big(STx_{n,}, TSx_n, TSx_n\big) = 0$.

Definition 1.7[5] A pair of self mappings S and T is said to be subsequentially continuous if $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \ in \ X$ such that

$\lim_{n\to\infty} STx_n = St$, $\lim_{n\to\infty} TSx_n = Tt$.

2. Main Result

In this section, we shall prove a common fixed point Theorem for four compatible and subsequentially continuous self maps in G-metric spaces.

Theorem 2.1. Let $A, B, S \ and \ T$ be four self mappings on a G-metric space(X, G), and suppose that the pairs (A, S) and (B, T) are compatible and subsequentially continuous (alternately subcompatible and reciprocally continuous) and satisfying the following inequality:

$$G(Ax, By, Bz) \ \le \ p\{G(Sx, Ty, Tz) \ + \ G(Ax, Sx, Sx)\} + q\ \{G(Sx, Ty, Tz) \ + \ G(By, Ty, Tz)\} + \mathrm{r}\ max\left\{G(Sx, Ty, Tz),\ \frac{G(Sx,By,Bz)+ G\big((Ax,Ty,Tz)\big)}{2}\right\}, \qquad (2.1)$$

where $p, q, r > 0 \ and \ p + q + r < 1$.

Then $A, B, S \ and \ T$ have a unique common fixed point in X.

Proof. Given that the pair (A, S) is sequentially continuous and compatible, so there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$ and $\lim_{n\to\infty} G\big(ASx_{n,}, SAx_n, SAx_n\big) = G(\mathrm{Az}, \mathrm{Sz}, \mathrm{Sz}) = 0$.

This implies that $Az = Sz$.

Thus z is a coincidence point of the pair (A, S).

Similarly, the pair (B, T) is sequentially continuous and compatible, so there exists a sequence $\{\mathrm{y_n}\}$ in X such that

$\lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = w$ for some $w \in X$ and

$\lim_{n\to\infty} G\big(BTy_{n,}, TBy_n, TBy_n\big) = G(\mathrm{Bw}, \mathrm{Tw}, \mathrm{Tw}) = 0$.

This implies that $Bw = Tw$, that is, w is a coincidence point of the pair (B, T).

Now, we claim that $z = w,\ if\ z \neq w$, then using the inequality (2.1) with $x = x_n, y = y_{n.} \ and \ z = y_n$, We have

$$G(A\mathrm{x_n}, By_n, By_n) \ \le \ p\{G(Sx_n, Ty_n, Ty_n) \ + \ G(Ax_n, Sx_n, Sx_n)\} + q\ \{G(Sx_n, Ty_n, Ty_n) \ + \ G(By_n, Ty_n, Ty_n)\} + r\ max\left\{G(Sx_n, Ty_n, Ty_n),\ \frac{G(Sx_n,By_n,By_n)+ G\big((A\mathrm{x_n},Ty_n,Ty_n)\big)}{2}\right\}.$$

Making $n \to \infty$, we get

$$G(z, w, w) \ \le \ p\{G(z, w, w) \ + \ G(z, z, z)\} + q\ \{G(z, w, w) \ + \ G(z, z, z)\}$$

$+ r\, max\left\{G(z,w,w),\ \frac{G(z,w,w)+G((z,w,w))}{2}\right\}$, that is,

$G(z,w,w) \le p\{G(z,w,w) + 0\} + q\{G(z,w,w) + 0\}$
$+ r\, max\{G(z,w,w),\ G(z,w,w)\}$, that is,

$G(z,w,w) \le p\{G(z,w,w)\} + q\{G(z,w,w)\}$
$+ r\{G(z,w,w)\}$
$< (p+q+r)\{G(z,w,w)\}$
$< \{G(z,w,w)\}$,

a contradiction.

Hence $z = w$.

Now, we prove that $Az = z$.

On the contrary, suppose that, $Az \ne z$.

On making use of the inequality (2.1) with $x = z, y = y_n, z = y_n$, we have

$G(Az, By_n, By_n) \le p\{G(Sz, Ty_n, Ty_n) + G(Az, Sz, Sz)\}$
$+q\{G(Sz, Ty_n, Ty_n) + G(By_n, Ty_n, Ty_n)\}$
$+ r\, max\left\{G(Sz, Ty_n, Ty_n),\ \frac{G(Sz,By_n,By_n)+G((Az,Ty_n,Ty_n))}{2}\right\}$.

Making limit as $n \to \infty$, we get

$G(Az,w,w) \le p\{G(Sz,w,w) + G(Az,Az,Az)\} + q\{G(Sz,w,w) + G(w,w,w)\}$
$+ r\, max\left\{G(Sz,w,w),\ \frac{G(Sz,w,w)+G((Sz,w,w))}{2}\right\}$, that is,

$G(Az,w,w) \le p\{G(Sz,w,w) + 0\} + q\{G(Sz,w,w) + 0\}$
$+ r\, max\{G(Sz,w,w),\ G(Sz,w,w)\}$, that is,

$G(Az,w,w) \le p\{G(Sz,w,w)\} + q\{G(Sz,w,w)\}$
$+ r\{G(Sz,w,w)\}$
$< (p+q+r)\{G(Sz,w,w)\}$
$< \{G(Az,w,w)\}$,

a contradiction.

Hence $Az = w = z$.

So $Az = z$.

Now, we claim that $Bz = z$.

Let, if possible, $Bz \ne z$.

Using the inequality (2.1) with $x = x_n,\ y = z$.

$G(Ax_n, Bz, Bz) \le p\{G(Sx_n, Tz, Tz) + G(Ax_n, Sx_n, Sx_n)\}$
$+q\{G(Sx_n, Tz, Tz) + G(Bz, Tz, Tz)\}$
$+ r\, max\left\{G(Sx_n, Tz, Tz),\ \frac{G(Sx_n,Bz,Bz)+G((Ax_n,Tz,Tz))}{2}\right\}$.

Making limit as $n \to \infty$, we get

$G(z,Bz,Bz) \le p\{G(Sz,Tz,Tz) + G(z,z,z)\} + q\{G(z,Tz,Tz) + G(Bz,Tz,Tz)\}$
$r\, max\left\{G(z,Tz,Tz),\ \frac{G(z,Bz,Bz)+G((z,Tz,Tz))}{2}\right\}$.

$G(z,Bz,Bz) \le p\{G(z,Bz,Bz) + 0\} + q\{G(z,Bz,Bz) + 0\}$
$+ r\, max\left\{G(z,Bz,Bz),\ \frac{G(z,Bz,Bz)+G((z,B,Bz))}{2}\right\}$.

$G(z,Bz,Bz) \le p\{G(z,Bz,Bz)\} + q\{G(z,Bz,Bz)\} + r\{G(z,Bz,Bz)\}$
$G(z,Bz,Bz) \le (p+q+r)\{G(z,Bz,Bz)\}$

$G(z, Bz, Bz) < G(z, Bz, Bz)$,
a contradiction.
Hence $Bz = z$.
So $Az = Bz = Sz = Tz = z$.
Hence z is a common fixed point of the four mappings $A, B, S\ and\ T$.
Uniqueness: Let w be another common fixed point of the mappings $A, B, S\ and\ T$. Then we have $Aw = Bw = Sw = Tw = w$. Now using the inequality (2.1) we have.

$$G(z,w,w) = G(\mathrm{Az},\mathrm{Bw},\mathrm{Bw}) \le p\{G(Sz,Tw,Tw) + G(Az,Sz,Sz)\}$$
$$+q\{G(Sz,Tw,Tw) + G(Bw,Tw,Tw)\}$$
$$+ r\, max\left\{G(Sz,Tw,Tw),\ \frac{G(Sz,Bw,Bw) + G\big((Az,Tw,Tw)\big)}{2}\right\}.$$
$$G(z,w,w) \le p\{G(z,w,w) + G(z,z,z)\} + q\{G(z,w,w) + G(w,w,w)\}$$
$$+ r\, max\left\{G(z,w,w),\ \frac{G(z,w,w) + G\big((z,w,w)\big)}{2}\right\}.$$
$G(z,w,w) \le p\{G(z,w,w)\} + q\{G(z,w,w)\} + r\, max\{G(z,w,w),\}$, that is,
$G(z,w.w) \le (p+q+r)\big(G(z,w,w)\big)$, that is,
$G(z,w.w) < \big(G(z,w,w)\big)$,
a contradiction. Hence z is a unique common fixed point of the four mappings $A, B, S\ and\ T$.
Now as our supposition the pair (A, S) is subcompatible and reciprocally continuous, then there exists a sequence $\{x_n\}$ in X such that
$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$ and
$\lim_{n\to\infty} G\big(ASx_n, SAx_n, SAx_n\big) = G(\mathrm{Az},\mathrm{Sz},\mathrm{Sz}) = 0$. This implies that $Az = Sz$. That is z is a coincidence point of the pair(A, S)
Similarly, as our supposition that the pair (B, T) is reciprocally continuous and subcompatible, then there exists a sequence $\{\mathrm{y_n}\}$ in X such that
$\lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = w$ for some $z \in X$ and $\lim_{n\to\infty} G\big(BTy_n, TBy_n, TBy_n\big) = G(\mathrm{Bw},\mathrm{Tw},\mathrm{Tw}) = 0$. This implies that $Bw = Tw$. That is w is a coincidence point of the pair(B, T). The remaining proof is as follows from the upper part.

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