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$\vec{P}_{2k}$ −Factorization of Complete Bipartite Symmetric Multi-digraph

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ABSTRACT

The graph P_k be the path graph on k vertices with $k-1$ edges and $\vec{P}_k$ be the directed path on k vertices with $k-1$ arcs. The graph $K_{m,n}$ be the complete bipartite graph with two partite sets X and Y having m and n elements respectively and the graph $\lambda K^*_{m,n}$ be the complete bipartite symmetric multi-digraph. In this paper we will discuss and established the conditions for the existence of the path decomposition of $\vec{P}_{2k}$ −factorization of complete bipartite symmetric multi-digraph. The following necessary and sufficient conditions for the existence of the path factorization of graphs are as follows, if $m=n$ and $m \equiv 0(mod\ 2k(2k-1)/d)$ then the graph $\lambda K^*_{m,n}$ has $\vec{P}_{2k}-$ factorization, where $d=$ gcd $(\lambda, 2(2k-1))$ and m, n, k, d and λ are positive integers.

Key words: Path Factorization, Bipartite Graph, Multi-digraph, Spanning subgraph.

1. INTRODUCTION

The problem on path factorization of complete bipartite graph was studied by Kazuhiko Ushio in his paper G-designs and related designs [1] and he gave the necessary and sufficient conditions for existence of path factorization of graph. The P_{2p} −Factorization of a complete bipartite graph completely studied by Hong Wang [2] and he gave the necessary and sufficient conditions for the existence of path factorization of complete bipartite graph $K_{m,n}$, after that the path factorization problem on multigraph was studied by Beiliang Du in P_{2k} −Factorization of a complete bipartite multigraphs [3] and he gave the necessary and sufficient conditions for P_{2k} −Factorization on complete bipartite multigraph. Expanding the work on path factorization the $\vec{P}_{2k} -$ Factorization of Complete bipartite symmetric digraphs [4], Bal Govind Shukla gave the necessary and sufficient conditions for the existence of $\vec{P}_{2k} -$ Factorization on$K^*_{m,n}$.

The complete bipartite graph $K_{m,n}$ have two partite sets of vertices X and Y such that order of each vertex i.e. $|X| = m$ and$|Y| = n$, where m and n are positive integers. The graph $K^*_{m,n}$ is the complete bipartite symmetric digraph with two partite sets X and Y contains symmetric edges of the directed graphs such that order of $|X| = m$ and $|Y| = n$, connected to m and n number of symmetric vertices. The graph $\lambda K_{m,n}$ be the complete bipartite multigraph which have λ times of graph $K_{m,n}$ such that each $K_{m,n}$ is isomorphic to each set of $\lambda K_{m,n}$. If $\lambda = 2$ then the complete bipartite multigraph $2K_{m,n}$ is isomorphic to complete bipartite symmetric digraph $K^*_{m,n}$. The graph $\lambda K^*_{m,n}$ is the complete bipartite symmetric multi-digraph is a graph in which arcs are directed graph. Here in the present paper we established the necessary and sufficient conditions for existence of $\vec{P}_{2k} -$ factorization of complete bipartite symmetric multi-digraphs $\lambda K^*_{m,n}$ such that, $m = n \; and \; m \equiv 0(mod\; 2k(2k - 1)/d)$, where $\lambda, m, n \; and \; k$ are positive integer and $d = \gcd(\lambda, 2(2k - 1))$.

2. MATHEMATICAL ANALYSIS

Here in the study of the path factorization i.e. $\vec{P}_{2k} -$ Factorization of Complete bipartite symmetric multi-digraphs, the $\vec{P}_{2k}$ be the directed path on $2k$ vertices and the graph $\lambda K^*_{m,n}$ be the symmetric complete bipartite multi-digraph. In the complete bipartite symmetric multi-digraph $\lambda K^*_{m,n}$ the each edge or arc is taken as λ times to complete bipartite symmetric digraph $K^*_{m,n}$. Now we prove the following theorems i.e. theorem 2.1 and 2.2 which are used in later for the necessary and sufficient conditions for the existence of $\vec{P}_{2k} -$ Factorization of Complete bipartite symmetric multi-digraphs$\lambda K^*_{m,n}$.

Theorem 2.1: If $\lambda K^*_{m,m}$ has $\vec{P}_{2k} -$ factorization then $\lambda K^*_{sm,sm}$ also have $\vec{P}_{2k} -$ factorization for any positive integer s.

Proof: The proof of above theorem can given by the construction. Consider a complete bipartite graph $K_{s,s}$ is one factorable [5], and the one factorization of it are $\{G_1, G_2, \dots, G_s\}$. Now for each i where$\{1 \le i \le s\}$, replace every arc of G_i with a $\lambda K^*_{m,m}$ to get a spanning subgraph H_i of $\lambda K^*_{sm,sm}$ such that the spanning subgraphH_i's $\{1 \le i \le s\}$ are pair wise arc-disjoint and hence there sum is $\lambda K^*_{sn,sn}$. Since the complete bipartite symmetric multi- digraph $\lambda K^*_{m,m}$ has $\vec{P}_{2k} -$ Factorable, therefore H_i is also $\vec{P}_{2k} -$ factorable. Hence $\lambda K^*_{sm,sm}$ is also $\vec{P}_{2k} -$ factorable.

Theorem 2.2: If $\lambda K^*_{m,m}$ has $\vec{P}_{2k} -$ factorization then $s\lambda K^*_{m,m}$ also has $\vec{P}_{2k} -$ factorization.

Proof: This theorem also proved by the construction. Construct a $\vec{P}_{2k} -$ factorization of $\lambda K^*_{m,m}$ and repeated same process in s times then we find that $s\lambda K^*_{m,m}$ also has $\vec{P}_{2k} -$ factorization.

The following theorem 2.3 gives necessary condition for the existence of $\vec{P}_{2k} -$ factorization of complete bipartite symmetric multi-digraph$\lambda K^*_{m,n}$.

Theorem 2.3: If complete bipartite symmetric multi-digraph $\lambda K^*_{m,n}$ has $\vec{P}_{2k} -$ factorization then $m = n$ and $m \equiv 0(mod 2k(2k - 1)/d)$, where λ, k, m and n are positive integers.

Proof: Since $\lambda K_{m,n}^*$ be the complete bipartite symmetric multi-digraph with two partite sets X and Y such that total number of vertices in X i.e. $|X| = m$ and the total number of vertices in Y i.e. $|Y| = n$. Let $\lambda K_{m,n}^*$ has $\vec{P}_{2k} -$ factor F and the total number of components of F are equal to t.

Hence $m = \frac{kt}{2} = n$ and $|F| = 2n(2k-1)/k$ is an integer which is independent on individual $\vec{P}_{2k} -$ factors. Hence for $m = n$,

$$m \equiv 0(mod\ k). \qquad \text{... (1)}$$

If the total number of components of $\vec{P}_{2k} -$ factors in $\vec{P}_{2k} -$ factorization of $\lambda K_{m,n}^*$ are b then $b = \frac{\lambda m^2}{2(2k-1)}$. If r be the total number of $\vec{P}_{2k}$ factors in $\vec{P}_{2k} -$ factorization of $\lambda K_{m,n}^*$then,

$$r = \frac{b}{t}$$

$$= \frac{\frac{\lambda m^2}{2(2k-1)}}{\frac{m}{k}}$$

$$= \frac{\lambda km}{2(2k-1)}$$

Where $\frac{\lambda km}{2(2k-1)}$ is a positive integer. Since $\gcd(k, 2k-1) = 1$, hence $\frac{\lambda m}{2(2k-1)}$ also be an integer and therefore,

$$\lambda m \equiv 0\big(mod\ 2(2k-1)\big).$$

Since $\lambda K_{m,n}^*$ isomorphic to $2\lambda K_{m,n}$ and $\gcd(\lambda, 2(2k-1)) = d$ then,

$$m \equiv 0(mod\ 2(2k-1)/d) \qquad \text{... (2)}$$

Now from equations (1) and (2), we have

$$m \equiv 0(mod\ 2k(2k-1)/d).$$

Now we consider a particular case of $\vec{P}_{2k} -$ factorization of $\lambda K_{m,n}^*$. The following fig. 2.1 and fig. 2.2 to 2.9 shown the $\vec{P}_2 -$ factor of complete bipartite symmetric multi-digraph $2K_{2,2}^*$,

here $k = 1, m = 2, n = 2$ and $\lambda = 2$

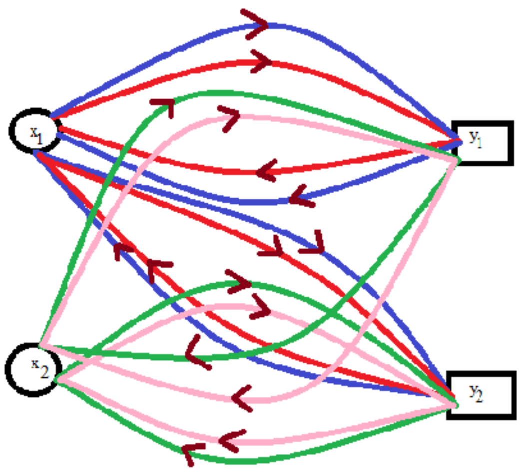

Fig. 2.1 Complete bipartite symmetric multi-digraph $2K^*_{2,2}$.

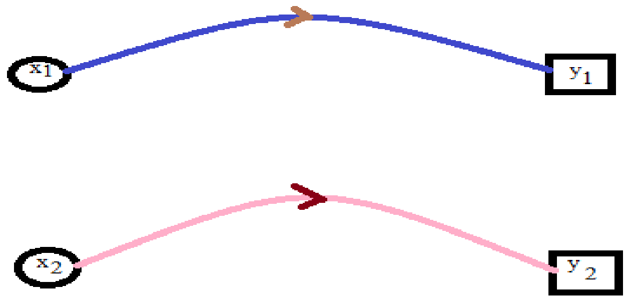

Fig. 2.2 Directed Path, x_1y_1 ; x_2y_2.

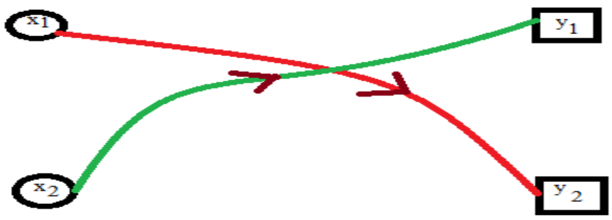

Fig. 2.3 Directed Path, x_1y_2 ; x_2y_1.

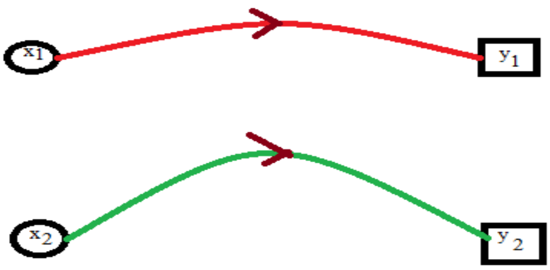

Fig. 2.4 Directed Path, x_1y_1 ; x_2y_2.

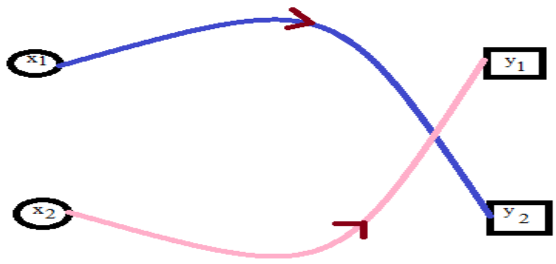

Fig. 2.5 Directed Path, x_1y_2 ; x_2y_2.

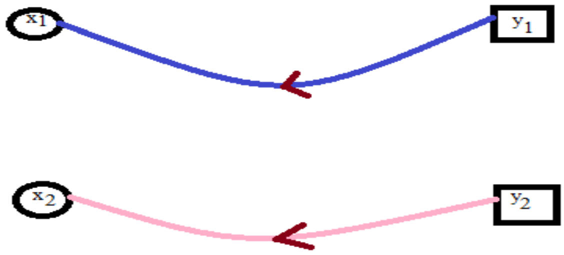

Fig 2.6 Directed Path, y_1x_1 ; y_2x_2.

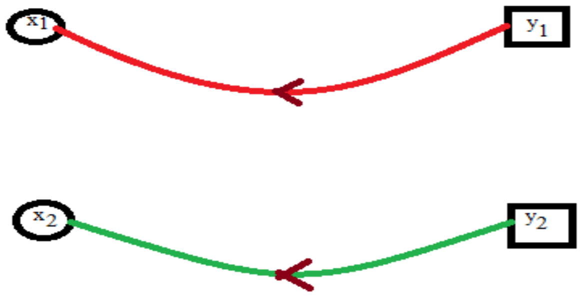

Fig. 2.7 Directed Path, y_1x_1 ; y_2x_2.

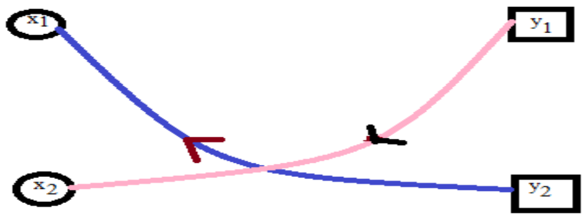

Fig. 2.8 Directed Path, y_2x_1 ; y_1x_2.

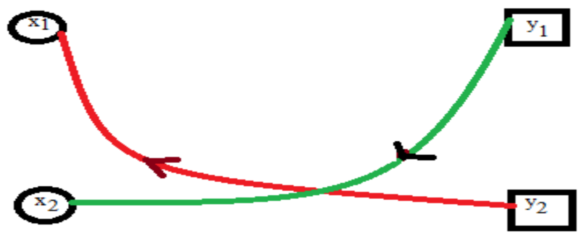

Fig. 2.9 Directed path y_2x_1 ; y_1x_2.

The following theorem 2.4 shown the sufficient conditions for the existence of $\vec{P}_{2k} -$ factorization of complete bipartite symmetric multi-digraph $\lambda K^*_{m,n}$.

Theorem 2.4: If $m = n$ and $m \equiv 0(mod\ 2k(2k-1)/d)$ then the graph $\lambda K^*_{m,n}$ has $\vec{P}_{2k} -$ factorization, where $d = \gcd(\lambda, 2(2k-1))$ and m, n, k, d and λ are positive integers.

Proof: It is given that $m = n$ and $m \equiv 0(mod\ k(2k-1)/d)$. Let $m = k(2k-1)s$ for some positive integer s. Hence from theorem 2.1 to show $\lambda K^*_{m,n}$ has $\vec{P}_{2k} -$ factorization it is only need to show that $\lambda K^*_{k(2k-1),k(2k-1)}$ has $\vec{P}_{2k} -$ factorization. Let the graph $\lambda K^*_{k(2k-1),k(2k-1)}$ has two partite sets X and Y such that

$$X = \{x_{i,j}: 1 \le i \le k, 1 \le j \le (2k-1)\},$$

$$Y = \{y_{i,j}: 1 \le i \le k, 1 \le j \le (2k-1)\}.$$

Now we are constructing $\vec{P}_{2k} -$ factor with a remark that the addition in the first subscripts of $x_{i,j}'s$ and $y_{i,j}'s$ is taken modulo k in $\{1, 2, \dots, k\}$and the second subscript is taken modulo $(2k-1)$ in $\{1, 2, \dots, (2k-1)\}$.

Now for each i with $1 \le i \le k$,

$$E_{2i-1} = \{x_{i,j}, y_{i,(j+2i-2)}: 1 \le j \le (2k-1)\},$$

$$E'_{2i-1} = \{y_{i,(j+2i-2)}, x_{i,j}: 1 \le j \le (2k-1)\}.$$

And for each i with $2 \le i \le k$,

$$E_{2i-2} = \{x_{i,j}, y_{(i-1),(j+2i-3)}: 1 \le j \le (2k-1)\},$$

$$E'_{2i-2} = \{y_{(i-1),(j+2i-3)}, x_{i,j}: 1 \le j \le (2k-1)\}.$$

Let the directed graph $\vec{F} = \cup_{1 \le i \le 2k-1}\{E_i, E'_i\}$then the directed graph $\vec{F}$ is a $\vec{P}_{2k} -$ factor of complete bipartite symmetric multi-digraph $\lambda K^*_{m,n}$. Define a bijection σ such that $\sigma: XUY \xrightarrow{onto} XUY$ and $\sigma(x_{i,j}) = x_{i+1,j}$ and $\sigma(y_{i,j}) = y_{i+1,j}$ where $1 \le i \le k$ and $1 \le j \le (2k-1)$.

For each $1 \le i, j \le k$, let

$$\overrightarrow{F_{i,j}} = \{\sigma^i(x)\sigma^j(y): x \in X, y \in Y\ and\ xy \in \vec{F}\}$$

It have shown that the digraph $\overrightarrow{F_{l,J}}$ is a $\vec{P}_{2k}-$ factor of complete bipartite symmetric digraph $K^*_{k(2k-1),k(2k-1)}$, hence their union is complete bipartite symmetric multi-digraph $\lambda K^*_{k(2k-1),k(2k-1)}$.

Hence using theorem 2.1 and theorem 2.2 we have seen that the graph $\lambda K^*_{m,n}$ has $\vec{P}_{2k}-$ factorization.

3. CONCLUSION

By applying theorem 2.3 and theorem 2.4 along with theorems 2.1-2.2, it can be seen that when $m = n$ and $m \equiv 0(mod\ 2k(2k-1)/d)$ then the graph $\lambda K^*_{m,n}$ have $\vec{P}_{2k}$ −factorization, where $d = gcd(\lambda, 2(2k-1))$.

4. REFRENCES

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