Turkish Online Journal of Qualitative Inquiry (TOJQI)
Volume 12, Issue 10, October 2021: 3994-4001

Necessity and Analysis of Model order Reduction Techniques

Anand Singh
Assistant Professor,
Bansal Institute of Engineering and Technology
Lucknow, India
anandsingh7777@gmail.com

Dr. Sushil Kumar Agrawal
Professor
Bansal Institute of Engineering and Technology
Lucknow, India
sushilagrawal1010@gmail.com

Abstract- There is a large number of techniques available for deriving reduced order models and lower order controllers. In this paper, the features and analysis of Frobenius Hankel (F H) norm, H2 norm and H∞ norm reduced order approximations is given. Also The necessary and sufficient conditions for the existence of an approximate solution within a specified error γ will be found, these conditions are given in terms of a set of linear matrix inequalities (LMI) and a matrix rank constraint for both continuous and discrete time multi input-multi output systems.
In this paper we will introduce some of the popular methods to reduce the complexity of models, which depends mainly on the balanced state space representation and the Hankel singular values. These methods are balanced truncation and Hankel norm reduction methods.
Keywords: Model Order Reduction, linear matrix inequalities, Modal Truncations.

I. INTRODUCTION

Simple models are preferred above complex models and accurate models are preferred above inaccurate models. To obtain high accuracy models, we usually need to implement complex models, while simple models are generally inaccurate. In this paper, we assume that a stable linear time-invariant system is given and we address the problem to approximate this system by a simpler one. The approximate system is required to have a dynamic behavior which is as close as possible, to the behavior of the system which we wish to approximate. The problem on this thesis is an optimal model approximation. This problem is definitely a relevant one as many models derived from first principles or identification routines tend to become complex. Also, in the design and synthesis of control systems, controllers may become too complex to be implemented.
The complexity of linear time-invariant models is generally defined as the dimension of the state vector of any minimal state space representation of the system. This number is also known as the McMillan degree or the order of the system. After the definition of complexity the model approximation problem can be stated as follows:
Given a stable, linear time-invariant system G(s) of McMillan degree n, find a lower to the behavior $\hat{B}$ of $\hat{G}(s)$.There are a large number of techniques available for deriving reduced order models and lower order controllers. One of the most commonly used methods is the balanced truncation method. The procedure is easy to be implemented and also the method is extensively studied [1]. Another method is the Hankel norm approximation [2]. As we can recognize from the model reduction techniques, there is an error between the original high order system and the obtained

reduced order model in some sense as an index of how good the approximate is. For both of the methods upper bounds on the error in the H∞ sense and also a lower bound for the Hankel norm approximation method are expressed in terms of the Hankel singular values of the original system. The previous methods do not in general produce optimal approximates in the H∞ sense and there are several methods for H∞ optimal model reduction are developed to reduce the error γ between the reduced and the original model [3, 4] are examples of the developed H∞ optimal model reduction.

II. STATE TRUNCATIONS

Consider a dynamical system in input-state-output form:

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = C\,x(t) + Du(t) \qquad (1.1)$$

Here, we have a system with n states, m inputs and p outputs. That is, $x(t) \in R^n$, $u(t) \in R^m$ and $y(t) \in Rp$ for all time instants $t \in R$.We will have (A_{11}, B_1, C_1, D) as a kth order truncation of (A, B, C, D). This kth order truncation for the system (1.1) is:

$$\dot{\xi}(t) = A_{11}\xi(t) + B_1u(t)$$
$$y(t) = C_1\,\xi(t) + Du(t) \qquad (1.2)$$

Although, the original system is stable, controllable and minimal the truncated system may not be.

III. MODAL TRUNCATIONS

Consider a state space transformation:

$$x = T\,x' \qquad (1.3)$$

for the system (1.1) with T a non-singular matrix of dimension n × n. Since such a transformation only amounts to rewriting the state variable in a new basis, this transformation does not affect the input-output behavior associated with (1.1).

Theorem 1.1. *The If* Σ *is represented by* (1.1)*, then the external (or input-output behavior) of* Σ *is equivalently represented by the input-state-output model.*

$$\dot{x}'(t) = T^{-1}AT\,x'(t) + T^{-1}Bu(t)$$
$$y(t) = C\,T\,x'(t) + Du(t) \qquad (1.4)$$

Proof: Can be obtained from (1.1) by substituting (1.3) in (1.1) and solving for x′. In fact, we describe all minimal input-state-output representations of Σ by varying T over the set of non-singular matrices. The transformation:

$$A \rightarrow T^{-1}AT = A'$$

is called a similarity transformation of the matrix A.
The characteristic polynomial of the A matrix occurring in (1.1) is the polynomial $p(s) = det(sI - A)$. We can write this polynomial in various formats.

$$p(s) = det(sI - A)$$
$$= p0 + p1s + ... + p_n s^n$$
$$= (s - \lambda 1)(s - \lambda 2)...(s - \lambda_n) \qquad (1.5)$$

where $\lambda 1, ..., \lambda n$ are the so called modes of the system. For the modal canonical form we assume that the natural frequencies $\lambda_1, \lambda_2, ...\lambda_n$ are all different. For each natural frequency λ_i there exists a (complex) eigenvector vi of dimension n such that $[\lambda_i - A]v_i = 0$. If we store these eigen vectors $v_1, ..., v_n$ in one n × n matrix.
$T = [v_1\ v_2\ ...\ v_n]$ then we obtain a non-singular transformation (1.3) and the transformed A matrix takes the form

$$A'x := T - 1AT = diag(\lambda_1, ..., \lambda_n) \qquad (1.6)$$

which is called a Jordan form of the matrix A. The resulting state space system is said to be in modal canonical form.

Definition 1.1: *(Modal canonical form) the input-state-output system:*

$$\dot{x}(t) = A'x(t) + B'u(t)$$

$$y(t) = C'x(t) + D'u(t) \qquad (1.7)$$

with $A' = T^{-1}AT$ as in (1.6), $B' = T^{-1}B$, $C' = TC$ and $D' = D$ is called a modal canonical state space representation.

Definition 1.2: *(Modal truncations) If (1.7) is a modal canonical state space system, then the k^{th} order truncation*

$$\dot{\xi}(t) = A^r 11\xi(t) + B^r 1u(t)$$

$$y(t) = C^r 1\,\xi(t) + D^r u(t) \qquad (1.8)$$

is called the k^{th} order modal truncation of (1.1).

IV. BALANCED TRUNCATION

A second popular procedure for model approximation is the method of balanced truncations. It requires a state truncation of a system which is represented in balanced state space form. The balanced state space representation is an input- state-output representation of the form (1.1) for which the controllability grammian and the observability grammian are equal and diagonal.

A. *Balanced state space representations*

Suppose that a minimal and stable state space representation (1.1) of a dynamical system is given. We define two matrices.

Since the system is assumed to be stable, the eigenvalues of A has a negative real part, and from this it follows that the integral in (1.9) is well defined. Note that P is an n × n real matrix, it is symmetric.

The observability grammian associated with the system (A, B, C, D) is the matrix

$$Q \triangleq \int_0^\infty e^{A^T t} C^T C e^{At} dt \qquad (1.10)$$

Again, the stability assumption implies that the integral in (1.10) is well defined. Q is an n × n real symmetric matrix.

Fortunately, to compute the controllability and observability grammians of a state space system, it is not necessary to perform the integration as in (1.9) and (1.10) the next theorem tell us how to obtain the grammians from the Lyapunov equation.

Theorem 1.2. *Given a minimal and stable system (1.1), its controllability gram- mian P is the unique positive definite solution of the Lyapunov equation.*

$$AP + PA^T + BB^T = 0 \qquad (1.11)$$

Similarly, the observability grammian Q is the unique positive definite solution of

$$A^T Q + QA + C^T C = 0 \qquad (1.12)$$

If the system we have is minimal, then grammians P *and* Q are the unique solutions to (1.11) and (1.12), respectively. The computation of the grammians is therefore equivalent to the algebraic problem to find solutions of Lyapunov equations (1.11) and (1.12). Balanced state space representations are now defined as follows.

Definition 1.3: *A minimal state space representation* (1.1) is *called balanced if the controllability and observability grammians are equal and diagonal, i.e, if*

$P = Q = diag(\sigma_1, \sigma_2, ...\sigma_n)$

Where, σ_i *is real and positive numbers that are ordered according to*

$\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_n > 0$

B. *Existence of Balanced State Space Representations*

To find the balanced representation of the system (1.1), let us assume that we cal- culated the controllability and observability grammians for the stable system (1.1) and let us see how these grammians transform if we change the basis of the state space. Thus, consider again the state space transformation (1.3). As we have seen, this results in the transformed state space parameters (A´, B´ , C´, D´) as shown on def- inition(1.1) yields that the transformed grammians take the form

$P' = T^{-1} P (T^{-1})^T$;

$Q' = T^T Q T$

This shows that the grammians depend strongly on the basis of the state space. However, their product

$\acute{P}\acute{Q} = T^{-1} P (T^{-1})^T T^T QT = T^{-1} PQT$

so that The eigenvalues of PQ are invariant under state space transformations. Let $\lambda_1, ..., \lambda_n$ denote the eigenvalues of the product PQ. Then λi are positive real numbers for $i = 1, ... n$ so that it makes sense to consider their square roots. We just showed that these numbers are system invariants: they do not change by transforming the basis of the state space. In the literature, these system invariants play a crucial role and are called the *Hankel singular values of the system (*1.1). To show that balanced state space representations actually exist, we need to construct a non-singular state transformation matrix T that simultaneously diagonalizes the controllability and the observability grammians P and Q.

The algorithm (which is of course implemented in MATLAB) is as follows:

INPUT: State space parameters (A, B, C, D) of a minimal, stable system of the form (1.1)

Step 1: Compute the grammians P and Q.

Step 2: Factorize P $= R>$ (the routine chol in MATLAB is doing this for you).

Step 3: Construct the matrix $RQR>$ and (since it is positive definite) factorize it as $RQR> = U>\Sigma^2 U$ where U is a unitary matrix (i.e., $UU> = U>U = I$) and $\Sigma = diag(\sigma_1, \sigma_2, ..., \sigma_n)$ then the numbers σ_i are the Hankel singular values (i.e., the square roots of the eigenvalues $\lambda_i(PQ)$).

Step 4: Define the non-singular matrix T: $= R>U\Sigma^{-1/2}$.

OUTPUT: the matrices $(\acute{A}, \acute{B}, \acute{C}, \acute{D})$ as defined in definition(1.1)

It is easily seen that the state transformation defined in step 4 of the algorithm achieves that the grammians of the transformed system are

$\acute{Q} = T^{>}QT = \Sigma^{-1/2} U^{>} RQR^{>} U \ \Sigma^{-1/2} = \Sigma^{-1/2}\Sigma^2\Sigma^{-1/2} = \Sigma\acute{P} = T^{-1} P (T^{-1})^{->} = \Sigma^{1/2} U^{>} R^{>} P R^{-1} U \Sigma^{1/2} = \Sigma^{1/2}\Sigma^{1/2} = \Sigma$

i.e., they are equal and diagonal with the Hankel singular values as diagonal elements. We thus proved the following important result.

Theorem 1.3. *Every stable dynamical system of the form (1.1) admits a balanced input-state-output representation.*

C. *Balance Truncation*

The above interpretation justifies the following definition of a model reduction procedure based on balanced state space representations.

Definition 1.4*(Balanced Truncations): If (1.1) is a stable, balanced state space system, then the k^{th} order truncation*

$$\dot{\xi}(t) = A_{11}\xi(t) + B_1 u(t)$$
$$y(t) = C_1\xi(t) + Du(t) \qquad (1.13)$$

This simple approximation method provides very efficient and good approximate models. It eliminates the poorly controllable and poorly observable states from a state space model. The number k may in practice be determined by inspecting the ordered sequence of Hankel singular values $\sigma_1, ..., \sigma_n$. A drop in this sequence (i.e., a number k for which $\sigma_{k+1}/\sigma_k << 1$) may give you a reasonable estimate of the order of a feasible approximate model. If $\sigma_k > \sigma_{k+1}$ (as will be the case in many practical situations) the k^{th} order balanced truncation turns out to have good properties.

Theorem 1.4. *Suppose that* (1.1) *is a balanced state space representation of a stable system. Let $k < n$ and suppose that $\sigma_k > \sigma_{k+1}$. Then the kth order balanced truncation is minimal, stable, balanced.*

Now let us consider the following remarks:

Remark 1.1: If $G(s)$ denotes the transfer function corresponding to (1.1) and $G_k(s)$ is the transfer function of a k^{th} order balanced truncation of $G(s)$ then it is known that the error $G - G_k$ satisfies,

$$\|G(s) - G_k(s)\|_\infty \leq 2(\sigma_{k+1} + \sigma_{k+2} + ... + \sigma_n) \qquad (1.14)$$

Thus the maximum peak in the Bode diagram of the error system is less than twice the sum of the tail of the Hankel singular values.

Remark 1.2: All the results of this section can be repeated for discrete time systems. Formulas change, but the ideas are identical.

Remark 1.3: In MATLAB the relevant routines for constructing balanced state space models are bal real for *continuous time systems and* dbalreal for *discrete time systems*.

V. HANKEL NORM REDUCTIONS

The Hankel norm reductions are among the most important techniques of model reduction procedures that exist today. It is one of the model approximation procedures that produce optimal approximate models according to some well-defined criterion that we will introduce below. It constitutes a beautiful theory associated with the names of Nehari, Arov-Adamjan-Krein (AAK) and Glover [2, 5]. Glover introduced state space ideas in this problem area and in our exposition we will follow his work.

A. *Hankel Singular Values and the Hankel Norm*

The Hankel norm of a system is easily computed. In fact, it turns out to be equal to the maximal Hankel singular value for the systems. For Discrete time systems is straightforward:

The controllability grammian is the positive definite matrix

$$P \triangleq \sum_{k=0}^{\infty} A^k B B^T (A^T)^k \qquad (1.15)$$

The observability grammian is the matrix

$$Q \triangleq \sum_{k=0}^{\infty}(A^T)^k C^T C A^k \qquad (1.16)$$

The grammians of the system are the unique positive definite solution to the Lyapunov equations

$$APA^T + BB^T - P = 0. \qquad (1.17)$$

$$A^TQA + C^TC - Q = 0. \qquad (1.18)$$

These equations form an efficient approach to solve for the grammians.

Definition 1.5: *The Hankel singular values of $G(s)$ ϵH_2 are given by*

$$\sigma_i(G(s)) \triangleq [\lambda_i(PQ)]^{\frac{1}{2}} \qquad (1.19)$$

Where P and Q are the controllability and observability grammians of $G(s)$.
As in the previous section, the eigenvalues $\lambda_1, ..., \lambda_n$ of the product PQ are input- output invariants and their square roots $\sigma_1, ..., \sigma_n$ are called the Hankel singular values. We assume that the Hankel singular values are ordered according to
$\sigma_1 \geq \sigma_2 \geq ..... \geq \sigma_n \geq 0$ and we obtain the following result.

Theorem 1.5. *If the system* Σ *is stable and represented by* (1.1), *then The Hankel norm*

$$\|\Sigma\|_H = \lambda_{max}^{1/2}(PQ) = \sigma_1.$$

Proof. The proof of this theorem cab is found on [2].
Thus the Hankel norm is nothing else than the largest Hankel singular value of the system and it can be computed directly from the product of the two grammians associated with a state space representation of the system. The same result holds for continuous and discrete time systems.

B. *The Hankel Norm Model Reduction Problem*

In the previous section we have seen how a balanced representation can lead to a reduced order model. However, this algorithm did not allow for an interpretation as an optimal approximation. That is, the model obtained by balanced truncation did not minimize a criterion in which we agreed how far the n^{th} order system Σ is apart from a k^{th} order approximation Σ_k . The Hankel-norm model reduction problem does involve such a criterion.
INPUT: the system (A, B, C, D) with (A, B) controllable (C, A) observable and A stable.

DESIRED: A system (A_k, B_k, C_k, D_k) of order $\leq k$ which approximates the sys- tem (A, B, C, D) optimal in the Hankel norm. Algorithm:
Step 1: Compute the Hankel singular values $\sigma_1, ..., \sigma_n$ of (A, B, C, D) and assume that

$\sigma_k > \sigma_{k+1} = \sigma_{k+2} = ... = \sigma_{k+r} > \sigma_{k+r+1} \geq ... \geq \sigma_n > 0$ i.e., σ_{k+r} has multiplicity r.
Step 2: Transform (A, B, C, D) to a partially balanced form

$$\mathbf{P} = \mathbf{Q} = \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix}.$$

where $\Sigma_1 = diag(\sigma_1, ..., \sigma_k, \sigma_{k+r+1}, ..., \sigma_n)$ and $\Sigma_2 = \sigma_{k+1} I_r$. That is, the $(k+1)^{st}$ Hankel singular value is put in the south-east corner of the joint gramians.
Step 3: Partition (A, B, C, D) conformably with the partitioned gramians as

$$\mathbf{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} C_1 & C_2 \end{pmatrix}$$

Further define

$$\Gamma = \Sigma_1^2 - \sigma_{k+1}^2 I.$$

and note that Γ is non-singular.

If $m \leq p$, proceed. If $m > p$, replace (A, B, C, D) by (A^T, C^T, B^T, D^T) and proceed.

Step 4: Determine a unitary matrix U satisfying $B_2 + C^T U = 0$.

Step 5: Determine the stable subsystem of $\hat{\Sigma}$ by choosing a basis of the state space of $\hat{\Sigma}$ such that

$$\hat{A} = \begin{pmatrix} \hat{A}_- & 0 \\ 0 & \hat{A}_+ \end{pmatrix}, \quad \hat{B} = \begin{pmatrix} \hat{B}_- \\ \hat{B}_+ \end{pmatrix}, \quad \hat{C} = \begin{pmatrix} \hat{C}_- & \hat{C}_+ \end{pmatrix}$$

Where A_- has all its eigen values in the open left half complex plane A_+ has all its eigen values in the open right half complex plane.

OUTPUT: Set

$A_k = \hat{A}$, $\quad B_k = \hat{B}$,

$C_k = \hat{C}$, $\quad D_k = \hat{D}$

Then the system Σ_k defined by

$$\frac{d\xi}{dt} = A_k \xi(t) + B_k u(t)$$
$$y(t) = C_k \xi(t) + D_k u(t)$$

is a state space representation of an optimal Hankel norm approximant of Σ and the error

$$\| \Sigma - \Sigma_k \|_H = \sigma_{k+1}$$

VI. Conclusion

There we see that there are number of approaches, such as [6, 7, 8, 9], use first order necessary conditions for optimality and develop optimization algorithms to find solutions to resulting nonlinear equations. Most of the methods in this direction are only applicable to the single input single output (SISO) case. Furthermore, it can be recognized from [10, 11] that whether the global optimum is always achievable is unclear in the continuous time case (while it is shown to exist in the discrete time case [12]) and that, in the case of nonexistence of the optimum, these approaches can only find local optima which may be far from the true (global) optimum. Even if the exis- tence of the global optimum is guaranteed, optimization methods based on search algorithms can have difficulties [13]. There may be one or more local optima and it is difficult to guarantee that the obtained solution is close to the global optimum. Moreover, there is usually no guarantee that the chosen stopping criterion for such a search algorithm is appropriate.

References

[1] S. Gugercin & A. C. Antoulas, *A survey of model reduction by balanced truncation and some new results*, International Journal of Control, 77(8) Page(s):748-766, (May 2004).

[2] K. Glover, All optimal Hankel-norm approximations of linear multivariable systems and their L1-error bounds, International Journal of Control, 39(6) Page(s):1115-1193, (June 1984).

[3] Y. Ebihara & T. Hagiwara, *On H1 model reduction using LMIs*, IEEE Transactions on Automatic Control, 49(7) Page(s):1187-1191,(July 2004).

[4] K. M. Grigoriadis, Optimal H1 model reduction via linear matrix inequalities: Continuous and discrete-time cases, Systems and Control Letters, 26(5) Page(s):321-333, (December 1995).

[5] V.M. Adamjan, D.Z. Arov, and M.G. Krein. Analytic Properties of the Schmidt Pairs of a Hankel operator and the generalized Schur-Takagi problem, Math. Sb. (N.S.), 86 Page(s):34-75, (1971).

[6] L. Baratchart, M. Cardelli & M. Olivi, *Identification and rational L*2 *approximation: A gradient algorithm*, MA Journal of Mathematical Control and In-formation, Automatica, 27(2) Page(s):413417, (March 1991).

[7] A. Ferrante, W. Krajewski, A. Lepschy, & U. Viaro, *Convergent algorithm forL*2 *model reduction*, Automatica, 35(1) Page(s):75-79,(January 1999).

[8] L. Meier III & D. G. Luenberger, *Approximation of linear constant systems*, IEEE Transactions on Automatic Control, 12(5) Page(s):585-588, (October1967).

[9] J. T. Spanos, M. H. Milman, & D. L. Mingori, *A new algorithm for L*2 *optimal model reduction*, Automatica, 28(5) Page(s):897-909,(September 1992).

[10] X.-X. Huang, W.-Y. Yan, & K. L. Teo, *On H*2 *near-optimal model reduction*, IEEE Transactions on Automatic Control, 46(8) Page(s):1279-1284,(August 2001).

[11] W. Yan & J. Lam, *An approximate approach to H*2 *optimal model reduction*, IEEE Transactions on Automatic Control, 44(7) Page(s):1341-1358,(July 1999).

[12] L. Baratchart, *Existence and generic properties of L*2 *approximants for linear systems*, MA Journal of Mathematical Control and Information, 3 Page(s):89-101,(1986).

[13] B. Hanzon & J. M. Maciejowski, *Constructive algebra methods for the L*2*- problem for stable linear systems*, Automatica, 32(12) Page(s):1645-1657, (December 1996).