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Research Article

Application of Little Fermat Theorem Solving some Problems about Division

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Abstract

This article provides solutions to some divisibility problems using Fermat's little theorem. To have beautiful solutions for each of those problems, mathematicians have combined knowledge of: Theory of divisibility and division with remainder, greatest common divisor, least common multiple, prime numbers, congruences, exponentiation, etc. This helps students think positively and flexibly about their existing knowledge and skills, and present concise and creative solutions.

Keywords: Divisibility, prime, positive integer, remainder.

Introduction

Solving an arithmetic problem using theorems has always been an attraction for every mathematician. In particular, arithmetic is one of the basic content in the high school math program and is featured in national and international competitions for excellent students and many other math competitions. Fermat's little theorem is one of the most famous and useful theorems in mathematics, which is applied in many different fields. The study of Fermat's little theorem, in addition to providing mathematical knowledge, also helps students to analyze, research, explore, exploit, develop problems to generalize, generalize knowledge and improve high thinking about problem solving. In addition, in-depth study of this theorem is also an effective tool to solve intensive problems of congruence, perfect squares, co-prime numbers, paired primes, etc., which are frequent topics. present on Math forums, Math meetings, Student Math Olympiads,... as well as related fields.

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Methodology

Authors use experiences, observations and examples with explanatory method to illustrate results and draw conclusion.

Author also use dialectical materialism methods.

Main Findings

1. Some Knowledge to Remember

• [1, p. 502] Fermat's Little Theorem: If p is a prime and a is a positive integer then

 $a^{p} \equiv a \pmod{p}$ Prove 1. Use induction by a. With a = 1 tense, the proposition is always true. Assume that the statement is true a I.e $p \mid a^{p} - a$ We will prove the statement to be true a + 1. Indeed: $(a+1)^{p} - (a+1) = (a^{p} - a) + \sum_{i=1}^{p-1} C_{p}^{k} a^{k}$ Using $p \mid C^{k}$ $(1 \leq k \leq p - 1)$ and assuming induction

Using $p | C_n^k$, $(1 \le k \le p-1)$ and assuming induction we deduce. Then $(a+1)^p \equiv (a+1) \pmod{p}$

So we complete the proof.

Prove 2.

Assume that (a, p) = 1 and it is necessary to prove that $a^{p-1} \equiv 1 \pmod{p}$

Consider integers a, 2a, ..., (p-1)a whose remainders when divided by p distinct (otherwise, with $ia \equiv ja \pmod{p}$ then $p \mid (i-j)a$ or $p \mid i-j$, sign "=" happens if i = j).

So
$$a, 2a, ..., (p-1)a \equiv 1.2...(p-1) \pmod{p}$$

It mean $a^{p-1}(p-1)! \equiv (p-1)! \pmod{p}$

So (p, (p-1)!) = 1 we infer what we have to prove.

Note. This theorem can be abbreviated as: $a^{p-1} \equiv 1 \pmod{p}$

• [2, p. 27] Definition of Divisibility

An integer a is said to be divisible by an integer b ($b \neq 0$) if there exists an integer x such that a = bx. Then we say that a is multiple of b or b is a divisor of a and denoted by $a \vdots b$ or $b \mid a$. If a is not divisible by b, we denote it $a \vdots b$ or $b \nmid a$, then we write a = bq + r and call r = 0, 1, 2, r - 1 the remainder in division a cho b.

• [3, p. 3] Properties of Divisibility

+) If
$$a \stackrel{!}{:} b$$
 then $ac \stackrel{!}{:} b$ with $\forall c \in Z$
+) If $a \stackrel{!}{:} b$ and $b \stackrel{!}{:} c$ then $a \stackrel{!}{:} c$
+) If $a \stackrel{!}{:} c$ and $b \stackrel{!}{:} c$ then $(ax+by) \stackrel{!}{:} c$ with all integer x, y
+) If $a \stackrel{!}{:} b$ and $a, b > 0$ then $a \ge b$
+) If $a \stackrel{!}{:} b$ and $b \stackrel{!}{:} a$ then $a = b$ or $a = -b$
+) If $m \ne 0$ and $a \stackrel{!}{:} b$ then $a = b \Leftrightarrow am \stackrel{!}{:} bm$

2. Some Practical Exercises

Exercise 1: Prove that $m = \frac{9^{47} - 1}{8}$ an odd composite number is not divisible by 3 and

$$3^{m-1} \equiv 1 \pmod{m}$$

The answer:

First, we show that m the composite number is odd. Indeed:

We have $m = \frac{9^{47} - 1}{8} = \frac{3^{47} - 1}{2} \cdot \frac{3^{47} + 1}{4}$

Since $\frac{3^{47}-1}{2}$, $\frac{3^{47}+1}{4}$ are all integers greater than 1, m is composite

Comment $m = \frac{9^{47} - 1}{8} = \frac{9^{47} - 1}{9 - 1} = 9^{46} + 9^{45} + \dots + 9 + 1 \equiv 1 \pmod{3}$, I.e *m* not divisible

by 3

Comment: 9^k ends in 9 if k is odd and ends in 1 if k is even So $9^{46} + 9^{45} 9^{46} + 9^{45} + ... + 9^2 + 9 = (9^{46} + 9^{45}) + (9^{44} + 9^{43}) + ... + (9^2 + 9)$ Obviously, each sum in brackets ends with the digit 0, and 46 is even, so the sum above consists of 23 pairs. Thus, m ends with the digit 1, so m is odd Now we will prove $3^{m-1} \equiv 1 \pmod{m}$

According to Fermat's little theorem, then $9^{47} \equiv 9 \pmod{47}$

As $9^{47} \equiv 9 \pmod{8}$ and (47, 8) = 1 then $9^{47} \equiv 9 \pmod{47.8}$

It means that $m - 1 = \frac{9^{47} - 9}{8} \div 47$

As 46 I seven number and (2;46) = 1 then $(m-1) \stackrel{!}{:} 94$ From that $(3^m - 1) \stackrel{!}{:} (2^{94} - 1) \Longrightarrow (3^m - 1) \stackrel{!}{:} m$ (do $(3^{2p} - 1) \stackrel{!}{:} 8m$ Or $3^{m-1} \equiv 1 \pmod{m}$

Exercise 2: With p = 5 is prime and a, b are two odd natural numbers such that $a + b \stackrel{!}{:} 5$ and $a - b \stackrel{!}{:} 4$. Prove $a^b + b^a \stackrel{!}{:} 10$

Tee answer:

Assume $a \ge b$

Call r is the remainder in division a by 5 then $a \equiv r \pmod{5} (1) (\operatorname{do} a + b \vdots 5)$

From

$$a \equiv r \pmod{5} \Rightarrow a + b \equiv r + b \pmod{5} \Rightarrow b + r \equiv 0 \pmod{5} \Rightarrow b \equiv -r \pmod{5}$$

$$a \equiv r \pmod{5} \Rightarrow a^b \equiv r^b \pmod{5}$$

$$b \equiv -r(\mod{5}) \Rightarrow b^a \equiv (-r)^a \pmod{5}$$
At that time $a^b + b^a \equiv r^b - r^a \pmod{5} \Rightarrow a^b + b^a \equiv r^b(1 - r^{a-b}) \pmod{5}$
On the other hand, by assumption $a - b \stackrel{!}{:} 4$ then $a - b \stackrel{!}{:} 4.k$
As r is not divisible by 5, so by Fermat's little theorem, we have:
 $a^b + b^a \equiv r^4 \equiv 1 \pmod{5} \Rightarrow r^{4.k} \equiv 1 \pmod{5} \Rightarrow r^{a-b} \equiv 1 \pmod{5}$
From that we get: $a^b + b^a \equiv 0 \pmod{5}$, I.e. $a^b + b^a \stackrel{!}{:} 5$
Furthermore: a^b , b^a are odd integers, so $a^b + b^a \stackrel{!}{:} 2$
Hence: $a^b + b^a \stackrel{!}{:} 10$

Exercise 3: Prove with $\forall k = 1, 2, 3, ..., 100$ then $C_{101}^k \vdots 101$

The answer:

Put
$$f(x) = \sum_{k=1}^{100} C_{101}^k x^k = (x+1)^{101} - (x^{101}+1)$$
 with $\forall x \in \mathbb{Z}$

Apply theorem little Fermat we have:

$$(x+1)^{101} \equiv x+1 \pmod{101}, \forall x \in \mathbb{Z}$$

On the other hand, according to theorem Fermat, we have: $x^{101} \equiv x \pmod{101}$, from that

 $x^{101} + 1 \equiv x + 1 \pmod{101}, \forall x \in \mathbb{Z}$

This leads to equation $f(x) = 0 \pmod{101}$ has 101 solutions.

As deg f = 100 then f not have level mod 101, i.e all coefficients of f divisible by

101. We have what to be proved.

Exercise 4: Prove with $\forall k = 1, 2, 3, ..., 36$ then $C_{36}^k \equiv (-1) \pmod{37}$

The answer:

Consider polynomial

$$f(x) = (x+1)^{36} - \frac{x^{37}+1}{x+1}$$

Comment: f mod's 37 rank is not over 35 or it has no rank

Apply theorem little Fermat we see equation $f(x) \equiv 0 \pmod{37}$ has 36 solutions mod 37 are 0, 1,..., 35. This means that all coefficients of f are divisible by 37

Hence $C_{36}^k \equiv (-1) \pmod{37}, 1 \le k \le 36$

Exercise 5: Prove with $\forall a, b \in \mathbb{Z}$ then $(ab^7 - ba^7) \stackrel{!}{:} 7$

The answer:

We have $ab^7 - ba^7 = ab(b^6 - a^6)$. Then:

+) If 7 | ab then $7 | ab(b^6 - a^6) = ab^7 - ba^7$

+) If $7 \setminus ab$ then as (7, a) = 1, (7, b) = 1 according to theorem little Fermat we have $b^7 \equiv b \pmod{7} \Longrightarrow b^6 \equiv 1 \pmod{7}$ (1)

Also according to theorem little Fermat we have $a^7 \equiv a \pmod{7} \Rightarrow a^6 \equiv 1 \pmod{7}$ (2) From(1) and (2) $\Rightarrow b^6 - a^6 \equiv 0 \pmod{7}$ Or $7 | b^6 - a^6 \Rightarrow 7 | ab^7 - ba^7$ So with $\forall a, b \in \mathbb{Z}$ then $(ab^7 - ba^7)$: 7

Exercise 6: Prove $37^{40} + 41^{36} - 1$: 37.41

The answer:

As 41 a prime, apply theorem little Fermat we have $37^{41} \equiv 37 \pmod{41}$.

From that $37^{41} - 37 \equiv 0 \pmod{41} \Longrightarrow 37(37^{40} - 1) \equiv 0 \pmod{41}$ (1)

As (37, 41) = 1 so from (1) we have: $37^{40} - 1 \equiv 0 \pmod{41}$ (2)

Obviously $41^{36} \equiv 30 \pmod{41}$ (3)

From (2), (3) we have $37^{40} + 41^{36} - 1 \equiv 0 \pmod{41}$

Because roles of 41, 37 the same we have: $41^{36} + 37^{40} - 1 \equiv 0 \pmod{37}$

As (37, 41) = 1 then $41^{36} + 37^{40} - 1 \equiv 0 \pmod{41.37}$

From that $37^{40} + 41^{36} - 1$: 37.41 (what to be proved)

Exercise 7: Prove $(3^{97} - 2^{97} - 1) \div 42.97$

The answer:

As 97 a prime according to theorem little Fermat we have:

 $3^{97} \equiv 3 \pmod{97}, 2^{97} \equiv 2 \pmod{97}$

So:

$$3^{97} - 2^{97} \equiv 1 \pmod{97} \Longrightarrow 3^{97} - 2^{97} - 1 \equiv 0 \pmod{97}$$

$$\Leftrightarrow (3^{97} - 3) - (2^{97} - 2) \equiv 0 \pmod{97}$$

$$\Leftrightarrow 3^{97} - 2^{97} - 1 \equiv 0 \pmod{97}$$

$$\Leftrightarrow (3^{97} - 2^{97} - 1) \vdots 97 (*)$$

On the other hand: $3^{97} - 2^{97} - 1 = [(3^{97} - 1) - 2^{97}] \div 2 (1)$

Then $3^{97} - 2^{97} - 1 \equiv -(-1)^{97} - 1 \equiv 0 \pmod{3}$, I.e $3^{97} - 2^{97} - 1 \stackrel{!}{\vdots} 3(2)$ Now we prove $(3^{97} + 2^{97} - 1) \stackrel{!}{\vdots} 17$ We have:

$$\begin{aligned} 3^{97} - 2^{97} - 1 &= 3.3^{96} - 2^{97} - 1 \\ &= 3.3^{2.48} - 2^{97} - 1 \\ &= 3.(3^2)^{48} - 2^{97} - 1 \\ &= 3.9^{48} - 2^{97} - 1 \\ &= 3.2^{48} - 2^{97} - 1 \pmod{7} \end{aligned}$$

And $3.2^{48} - 2^{97} - 1 &= (2+1).2^{48} - 2^{97} - 1 &= 2^{49} + 2^{48} - 2^{97} - 1 \end{aligned}$
So:
$$3^{97} - 2^{97} - 1 &\equiv 2^{49} + 2^{48} - 2^{97} - 1 \pmod{7} \\ &\equiv 2^{49} + (2^3)^{16} - 2^{97} - 1 \pmod{7} \\ &\equiv 2^{49} + 8^{16} - 2^{97} - 1 \pmod{7} \\ &\equiv 2^{49} - 2^{97} \pmod{7} \\ &\equiv 2^{49} - 2^{97} \pmod{7} \\ &\equiv -(2^{97} - 2^{49}) \pmod{7} \\ &\equiv -(2^{49} (2^{48} - 1)) \pmod{7} \\ &\equiv -(2^{49} (8^{16} - 1) \pmod{7}) \end{aligned}$$

 $\equiv 0 \pmod{7} \quad (3)$

From (1), (2), (3) we have $(3^{97} - 2^{97} - 1) \div 42$ (**) From (*) and ((**) we have: $(3^{97} - 2^{97} - 1) \div 42.97$

Exercise 8: Find all positive integers n such that $2^n - 1 \stackrel{?}{:} 7$ The answer:

As 2 a prime according to theorem little Fermat, we have $2^7 \equiv 2 \pmod{7}$

From that $2^6 \equiv 1 \pmod{7} \Rightarrow 2^6 - 1 \equiv 0 \pmod{7}$, and $2^6 - 1 \equiv (2^3 - 1)(2^3 + 1)$, so $7 \mid (2^3 - 1)(2^3 + 1) \Rightarrow 7 \mid 2^3 + 1 \Leftrightarrow 2^3 \equiv 1 \pmod{7}$

So all numbers n divisible by 3 satisfying requirement.

Exercise 9: Prove that there are infinitely many positive integers n that satisfy $(2^n - 1)$: 67 The answer:

As 2 a prime according to theorem little Fermat, we have $2^{67} \equiv 2 \pmod{67}$

From that $2^{66} \equiv 1 \pmod{67} \Longrightarrow 2^{m.66} \equiv 1 \pmod{67}$, with *m* positive integer.

Take n = m.66, with $m \equiv -1 \pmod{67}$, we have $n = m.66 \equiv 1 \pmod{67}$

And $2^n - n \equiv 2^n - 1 \equiv 0 \pmod{67}$

So there are infinitely many positive integers m such that $m \equiv -1 \pmod{67}$ there are infinitely many positive integers n satisfying $(2^n - 1) \stackrel{.}{:} 67$

Exercise 10: [5, tr. 45] Prove $3^{100} - 3 \div 13$

The answer:

As 13 a prime, according to theorem Fermat we have:

 $3^{12} \equiv 1 \pmod{13}$

With 100 = 12.8 + 4 so $3^{100} = (3^{12})^8 \cdot 3^4 \equiv 3^4 \pmod{13}$

But $3^4 \equiv 81 \equiv 3 \pmod{13}$

Then the remainder in division $3^{100} - 3$ by 13 is 0, or $3^{100} - 3$:13

Exercise 11: With $n \ge 2$, $a \ge 0$ is a positive integer and m is a prime such that $a^m \equiv 1 \pmod{m^n}$. Prove if m > 2 then $a \equiv 1 \pmod{m^{n-1}}$ and if m = 2 then $a \equiv \pm 1 \pmod{2^{n-1}}$

The answer:

We have $a^m \equiv 1 \pmod{m^n}$ with $n \ge 2$, so $a^m \equiv 1 \pmod{m}$

From theorem little Fermat $a^m \equiv a \pmod{m}$, so $a \equiv 1 \pmod{m}$

In case a = 1, Obviously there is something to prove

If $a \neq 1$, we put $a = 1 + km^{u}$ (here $u \ge 1$ and $k \searrow m$. So, m > 2, $a^{m} = 1 + km^{u+1} + Vm^{2u+1}$ with V an integer Hence m + 1 > n so $a \equiv 1 \pmod{m^{n-1}}$

In case m=2, we have $2^n | a^2 - 1 = (a-1)(a+1)$

As $a-1 \neq 2$, $a+1 \neq 2$ so a-1, a+1 both cannot be multiples of 4. So either expression a-1, a+1 is divisible by 2^{n+1} , I.e $a \equiv \pm 1 \pmod{2^{n-1}}$. We have what to be proved **Exercise 12:** With a as integer, knowing $a^{25} - a \stackrel{.}{:} m (m > 1)$. Find m

The answer:

Suppose to find a number m that satisfies the requirements of the problem. Then with p is a prime number, because p^2 is not divisible by $p^{25} - p$, so we have p^2 not divisible by m. So m multiples of distinct primes

On the other hand, we have: $a^{25} - 2 = 2.3^2 \cdot 5.7 \cdot 13 \cdot 17 \cdot 241$

As $3^{25} \equiv -3 \pmod{17}$ and $3^{25} \equiv 32 \pmod{241}$ so *m* not divisible by 17 and 241.

According to theorem little Fermat, we have $a^{25} \equiv a \pmod{p}$ when $p = \{2; 3; 5; 7; 13\}$ Thus, m will be equal to the divisors of: 2. 3. 5. 7. 13, different from 1 and have $2^5 - 1 = 31$ their divisors.

Exercise 13: Let t be a positive integer, k be an odd natural number and $m = k \cdot 2^t + 1$ an odd prime number. Assume that for x, y are natural numbers satisfying $x^{2^t} + y^{2^t} \\\vdots m$. Prove that then x, y simultaneously divisible by m.

The answer:

We prove it by the method of contradiction

Assume the opposite, $x \ m$, then $y \ m$

As *m* a prime according to theorem little Fermat we have: $x^{m-1} \equiv 1 \pmod{m}, y^{m-1} \equiv 1 \pmod{m}.$

As $m-1=k.2^t$ then we have: $x^{k.2^t} \equiv 1 \pmod{m}$, $y^{k.2^t} \equiv 1 \pmod{m}$

From that we get $x^{k.2^{t}} + y^{k.2^{t}} \equiv 2 \pmod{m}$. (1)

According to assumption $x^{2'} + y^{2'} \\ \vdots \\ m \\ \Leftrightarrow x^{2'} + y^{2'} \equiv 0 \pmod{m}$

Because k odd natural number:

 $x^{k,2^{t}} + y^{k,2^{t}} = (x^{2^{t}})^{k} + (y^{2^{t}})^{k} \vdots (x^{2^{t}} + y^{2^{t}}) \equiv 0 \pmod{m} (2)$

We see (1) and (2) contradict. So x and y at the same time divisible by m (what to be proved)

Exercise 14: Using theorem Fermat find remainder when divide 3456^{789} by 23

The answer:

According to theorem Fermat, we get:

 $3456^{789} \equiv (3456^{789})^{35} . 3456^{19}$

$$\equiv 1^{35} \cdot 6^{19} \pmod{23}$$

$$\equiv 6^{19} \pmod{23}$$

$$\equiv 6^{18} \cdot 6 \pmod{23}$$

$$\equiv 6^{3.6} \cdot 6 \pmod{23}$$

$$\equiv (6^3)^6 \cdot 6 \pmod{23}$$

$$\equiv (9^2)^3 \cdot 6 \pmod{23}$$

$$\equiv 12^3 \cdot 6 \pmod{23}$$

$$\equiv 3 \cdot 6 \pmod{23}$$

$$\equiv 18 \pmod{23}$$

So remainder when divide 3456^{789} by 23 is 18

Exercise 15: Prove $1^{18} + 2^{18} + 3^{18} + 4^{18} + 5^{18} + 6^{18}$: 7

The answer:

The numbers 1; 2; 3; 4; 5; 6 is a co-prime to 7, so by Fermat's theorem we have the following congruences:

$$1^{6} \equiv 1 \pmod{7};$$

 $2^{6} \equiv 1 \pmod{7};$
 $3^{6} \equiv 1 \pmod{7};$
 $6^{6} \equiv 1 \pmod{7};$

Raising to the third power on both sides of the above congruence, we get:

 $1^{18} \equiv 1 \pmod{7}$ $2^{18} \equiv 1 \pmod{7}$

 $3^{18} \equiv 1 \pmod{7}$

.....

 $6^{18} \equiv 1 \pmod{7}$

Adding the sides of those congruent, we get:

$$1^{18} + 2^{18} + 3^{18} + 4^{18} + 5^{18} + 6^{18} \equiv 0 \pmod{7}$$
$$\Leftrightarrow 1^{18} + 2^{18} + 3^{18} + 4^{18} + 5^{18} + 6^{18} \vdots 7$$

Exercise 16: With *m* is prime and m > 5. Prove $m^8 - 1 \stackrel{!}{:} 240$

The answer:

Analyze 240 Product analysis of prime factors (standard form), we get $240 = 2^4.3.5$ Apply little Fermat theorem, we have $m^4 \equiv 1 \pmod{5}$ and $m^2 \equiv 1 \pmod{3}$ Call *a* positive integer, we see $(a, 2^4) = 1$ when *a* is odd.

On the other hand early formula q(m) = m(1 - 1)(1 - 1) = (1 - 1) is the

On the other hand apply formula $\varphi(m) = m(1 - \frac{1}{p_1})(1 - \frac{1}{p_2})....(1 - \frac{1}{p_k})$ is the number of smaller than m and prime positive numbers with m (vói $m = p_1^{m_1}.p_2^{m_2}....p_k^{m_k}$ is the

standard analysis of *m*), we get $\varphi(24) = \varphi(2^3.3) = 24.(1-\frac{1}{2}).(1-\frac{1}{3}) = 8 = 2^3$

According to theorem Euler, we get $m^8 \equiv 1 \pmod{16}$.

Hence $m^8 \equiv 1 \pmod{p}$ with p = 3, 5, 16. From that $m^8 - 1 \stackrel{!}{\cdot} 240$ (what to be proved) **Exercise 17:** With p as a prime and $p - 2 \stackrel{!}{\cdot} 3$. Prove with a, b are integers satisfying $a^3 - b^3 \stackrel{!}{\cdot} p$ then $a - b \stackrel{!}{\cdot} p$

The answer:

Consider 2 cases: a or b divisible by p and a, b are not divisible by p

- +) Case 1: If $a \colon p \Longrightarrow b^3 \colon p \Longrightarrow b \colon p$, so $a b \colon p$
- +) Case 2: If a, b are not divisible by p

As assumption $p-2:3 \Rightarrow p=3k+2 \ (k \in \Box)$

According to little Fermat theorem, we get $a^{p-1} \equiv 1 \pmod{p}$

But p = 3k + 2 ($k \in \Box$) then $a^{3k+1} \equiv 1 \pmod{p}$

Similar proof, we get $b^{3k+1} \equiv 1 \pmod{p}$

From that $a^{3k+1} - b^{3k+1}$: p

According to assumption $a^3 - b^3 \vdots p \Longrightarrow a^{3k} - b^{3k} \vdots p$

From that a-b: *p* (what to be proved)

Conclusion

The diverse and complex topics of Arithmetic problems are always attractive to good students, because solving each problem in this field helps train and develop students' thinking. Using Fermat's little theorem to solve divisibility problems, in addition to providing quick, concise and accurate solutions, also stimulates new discoveries and discoveries in problem solving, helping learners to be creative in learning. Through this type of math, learners have the ability to think deeply, the ability to flexibly apply knowledge content to each type of exercise accordingly, so that they can achieve the best effect to meet the program. new general education with the goal of orienting student capacity development.

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