Turkish Online Journal of Qualitative Inquiry (TOJQI)
Volume 12, Issue 10, October 2021: 5046-5059

Research Article

Application of Little Fermat Theorem Solving some Problems about Division

Nguyen Tuyet Nga [1]

Abstract

This article provides solutions to some divisibility problems using Fermat's little theorem. To have beautiful solutions for each of those problems, mathematicians have combined knowledge of: Theory of divisibility and division with remainder, greatest common divisor, least common multiple, prime numbers, congruences, exponentiation, etc. This helps students think positively and flexibly about their existing knowledge and skills, and present concise and creative solutions.

Keywords: *Divisibility, prime, positive integer, remainder.*

Introduction

Solving an arithmetic problem using theorems has always been an attraction for every mathematician. In particular, arithmetic is one of the basic content in the high school math program and is featured in national and international competitions for excellent students and many other math competitions. Fermat's little theorem is one of the most famous and useful theorems in mathematics, which is applied in many different fields. The study of Fermat's little theorem, in addition to providing mathematical knowledge, also helps students to analyze, research, explore, exploit, develop problems to generalize, generalize knowledge and improve high thinking about problem solving. In addition, in-depth study of this theorem is also an effective tool to solve intensive problems of congruence, perfect squares, co-prime numbers, paired primes, etc., which are frequent topics. present on Math forums, Math meetings, Student Math Olympiads,... as well as related fields.

[1] Tan Trao University, Tuyen Quang, Vietnam, tuyetngasp@gmail.com

Received: xxxxxxx, Accepted: xxxxxxx

Methodology

Authors use experiences, observations and examples with explanatory method to illustrate results and draw conclusion.

Author also use dialectical materialism methods.

Main Findings

1. Some Knowledge to Remember

- **[1, p. 502] Fermat's Little Theorem: If p is a prime and a is a positive integer then** $a^p \equiv a (\mod p)$

Prove 1.

Use induction by a.

With a = 1 tense, the proposition is always true.

Assume that the statement is true a I.e $p \mid a^p - a$

We will prove the statement to be true $a+1$. ***Indeed:***

$$(a+1)^p - (a+1) = (a^p - a) + \sum_{i=1}^{p-1} C_p^k a^k$$

Using $p \mid C_n^k, (1 \le k \le p-1)$ ***and assuming induction we deduce. Then***

$$(a+1)^p \equiv (a+1) \ (\mod p)$$

So we complete the proof.

Prove 2.

Assume that $(a,\ p) = 1$ ***and it is necessary to prove that*** $a^{p-1} \equiv 1 \, (\mod p)$

Consider integers $a, 2a, \ldots, (p-1)a$ ***whose remainders when divided by p distinct (otherwise, with*** $ia \equiv ja \ (\mod p)$ then $p \mid (i-j)a$ or $p \mid i-j$, sign "$=$" happens if $i = j$).

So $a, 2a, \ldots, (p-1)a \equiv 1.2....(p-1) \ (\mod p)$

It mean $a^{p-1}(p-1)! \equiv (p-1)! \ (\mod p)$

So $(p, (p-1)!) = 1$ ***we infer what we have to prove.***

Note. This theorem can be abbreviated as: $a^{p-1} \equiv 1 \, (\mod p)$

- **[2, p. 27] Definition of Divisibility**

An integer a is said to be divisible by an integer b $(b \neq 0)$ if there exists an integer x such that $a = bx$. Then we say that a is multiple of b or b is a divisor of a and denoted by $a \vdots b$ or $b \mid a$. If a is not divisible by b, we denote it $a \not\vdots b$ or $b \nmid a$, then we write $a = bq + r$ and call $r = 0, 1, 2, r-1$ the remainder in division a cho b.

- **[3, p. 3] Properties of Divisibility**

+) If $a \vdots b$ then $ac \vdots b$ with $\forall c \in Z$

+) If $a \vdots b$ and $b \vdots c$ then $a \vdots c$

+) If $a \vdots c$ and $b \vdots c$ then $(ax+by) \vdots c$ with all integer x, y

+) If $a \vdots b$ and $a, b > 0$ then $a \geq b$

+) If $a \vdots b$ and $b \vdots a$ then $a = b$ or $a = -b$

+) If $m \neq 0$ and $a \vdots b$ then $a = b \Leftrightarrow am \vdots bm$

2. Some Practical Exercises

Exercise 1: ***Prove that*** $m = \frac{9^{47}-1}{8}$ ***an odd composite number is not divisible by 3 and*** $3^{m-1} \equiv 1 \pmod{m}$

The answer:

First, we show that m ***the composite number is odd. Indeed:***

We have $m = \frac{9^{47}-1}{8} = \frac{3^{47}-1}{2} . \frac{3^{47}+1}{4}$

Since $\frac{3^{47}-1}{2}, \frac{3^{47}+1}{4}$ ***are all integers greater than 1, m is composite***

Comment $m = \frac{9^{47}-1}{8} = \frac{9^{47}-1}{9-1} = 9^{46} + 9^{45} + ... + 9 + 1 \equiv 1 \pmod{3}$, I.e m not divisible by 3

Comment: 9^k ***ends in 9 if k is odd and ends in 1 if k is even***

So $9^{46} + 9^{45}$ $9^{46} + 9^{45} + ... + 9^2 + 9 = (9^{46} + 9^{45}) + (9^{44} + 9^{43}) + ... + (9^2 + 9)$

Obviously, each sum in brackets ends with the digit 0, and 46 is even, so the sum above consists of 23 pairs. Thus, m ends with the digit 1, so m is odd

Now we will prove $3^{m-1} \equiv 1 \pmod{m}$

According to Fermat's little theorem, then $9^{47} \equiv 9 \pmod{47}$

As $9^{47} \equiv 9 \pmod{8}$ and $(47, 8) = 1$ then $9^{47} \equiv 9 \pmod{47.8}$

It means that $m-1 = \dfrac{9^{47}-9}{8} \vdots 47$

As 46 I seven number and $(2;46) = 1$ then $(m-1) \vdots 94$

From that $(3^m - 1) \vdots (2^{94} - 1) \Rightarrow (3^m - 1) \vdots m$ (do $(3^{2p} - 1) \vdots 8m$

Or $3^{m-1} \equiv 1 \pmod{m}$

Exercise 2: With $p = 5$ is prime and a,b are two odd natural numbers such that $a+b \vdots 5$ and $a-b \vdots 4$. Prove $a^b + b^a \vdots 10$

Tee answer:

Assume $a \geq b$

Call r is the remainder in division a by 5 then $a \equiv r \pmod{5}$ (1) (do $a+b \vdots 5$)

From

$a \equiv r \pmod{5} \Rightarrow a+b \equiv r+b \pmod{5} \Rightarrow b+r \equiv 0 \pmod{5} \Rightarrow b \equiv -r \pmod{5}$

$a \equiv r \pmod{5} \Rightarrow a^b \equiv r^b \pmod{5}$

$b \equiv -r \pmod{5} \Rightarrow b^a \equiv (-r)^a \pmod{5}$

At that time $a^b + b^a \equiv r^b - r^a \pmod{5} \Rightarrow a^b + b^a \equiv r^b(1 - r^{a-b}) \pmod{5}$

On the other hand, by assumption $a-b \vdots 4$ then $a-b \vdots 4.k$

As r is not divisible by 5, so by Fermat's little theorem, we have:

$a^b + b^a \equiv r^4 \equiv 1 \pmod{5} \Rightarrow r^{4.k} \equiv 1 \pmod{5} \Rightarrow r^{a-b} \equiv 1 \pmod{5}$

From that we get: $a^b + b^a \equiv 0 \pmod{5}$, I.e $a^b + b^a \vdots 5$

Furthermore: a^b, b^a are odd integers, so $a^b + b^a \vdots 2$

Hence: $a^b + b^a \vdots 10$

Exercise 3: Prove with $\forall k = 1, 2, 3, ..., 100$ then $C_{101}^k \vdots 101$

The answer:

Put $f(x)=\sum_{k=1}^{100}C_{101}^{k}x^{k}=(x+1)^{101}-(x^{101}+1)$ with $\forall x\in Z$

Apply theorem little Fermat we have:

$(x+1)^{101}\equiv x+1\,(\text{mod}\,101)\,,\forall x\in Z$

On the other hand, according to theorem Fermat, we have: $x^{101}\equiv x(\text{mod}\,101)$, from that

$x^{101}+1\equiv x+1\,(\text{mod}\,101),\ \forall x\in Z$

This leads to equation $f(x)=0\ (\text{mod}\,101)$ has 101 solutions.

As $\deg\ f=100$ then f not have level $\text{mod}\,101$, i.e all coefficients of f divisible by 101. We have what to be proved.

Exercise 4: Prove with $\forall k=1,\ 2,\ 3,...,\ 36$ then $C_{36}^{k}\equiv(-1)(\text{mod}\,37)$

The answer:

Consider polynomial

$$f(x)=(x+1)^{36}-\frac{x^{37}+1}{x+1}$$

Comment: f mod's 37 rank is not over 35 or it has no rank

Apply theorem little Fermat we see equation $f(x)\equiv 0\ (\text{mod}\,37)$ has 36 solutions mod 37 are $0,1,...,\ 35$. This means that all coefficients of f are divisible by 37

Hence $C_{36}^{k}\equiv(-1)(\text{mod}\,37),1\le k\le 36$

Exercise 5: Prove with $\forall a,b\in Z$ then $(ab^{7}-ba^{7})\,\vdots\,7$

The answer:

We have $ab^{7}-ba^{7}=ab(b^{6}-a^{6})$. Then:

+) If $7\,|\,ab$ then $7\,|\,ab(b^{6}-a^{6})=ab^{7}-ba^{7}$

+) If $7\nmid ab$ then as $(7,\,a)=1,\,(7,\,b)=1$ according to theorem little Fermat we have

$b^{7}\equiv b\ (\text{mod}\ 7)\Rightarrow b^{6}\equiv 1\ (\text{mod}\ 7)$ (1)

Also according to theorem little Fermat we have $a^{7}\equiv a\ (\text{mod}\ 7)\Rightarrow a^{6}\equiv 1\ (\text{mod}\ 7)$ (2)

From(1) and (2)$\Rightarrow b^{6}-a^{6}\equiv 0\ (\text{mod}\ 7)$

Or $7\,|\,b^{6}-a^{6}\Rightarrow 7\,|\,ab^{7}-ba^{7}$

So with $\forall a,b \in Z$ then $(ab^7 - ba^7) \vdots 7$

Exercise 6: Prove $37^{40} + 41^{36} - 1 \vdots 37.41$

The answer:

As 41 a prime, apply theorem little Fermat we have $37^{41} \equiv 37 \pmod{41}$.

From that $37^{41} - 37 \equiv 0 \pmod{41} \Rightarrow 37(37^{40} - 1) \equiv 0 \pmod{41}$ (1)

As $(37, 41) = 1$ so from (1) we have: $37^{40} - 1 \equiv 0 \pmod{41}$ (2)

Obviously $41^{36} \equiv 30 \pmod{41}$ (3)

From $(2), (3)$ we have $37^{40} + 41^{36} - 1 \equiv 0 \pmod{41}$

Because roles of $41, 37$ the same we have: $41^{36} + 37^{40} - 1 \equiv 0 \pmod{37}$

As $(37, 41) = 1$ then $41^{36} + 37^{40} - 1 \equiv 0 \pmod{41.37}$

From that $37^{40} + 41^{36} - 1 \vdots 37.41$ (what to be proved)

Exercise 7: Prove $(3^{97} - 2^{97} - 1) \vdots 42.97$

The answer:

As 97 a prime according to theorem little Fermat we have:

$3^{97} \equiv 3 \pmod{97}$, $2^{97} \equiv 2 \pmod{97}$

So:

$$3^{97} - 2^{97} \equiv 1 \pmod{97} \Rightarrow 3^{97} - 2^{97} - 1 \equiv 0 \pmod{97}$$
$$\Leftrightarrow (3^{97} - 3) - (2^{97} - 2) \equiv 0 \pmod{97}$$
$$\Leftrightarrow 3^{97} - 2^{97} - 1 \equiv 0 \pmod{97}$$
$$\Leftrightarrow (3^{97} - 2^{97} - 1) \vdots 97 \ (*)$$

On the other hand: $3^{97} - 2^{97} - 1 = [(3^{97} - 1) - 2^{97}] \vdots 2$ (1)

Then $3^{97} - 2^{97} - 1 \equiv -(-1)^{97} - 1 \equiv 0 \pmod{3}$, I.e $3^{97} - 2^{97} - 1 \vdots 3$ (2)

Now we prove $(3^{97} + 2^{97} - 1) \vdots 17$

We have:

$$3^{97}-2^{97}-1=3.3^{96}-2^{97}-1$$
$$=3.3^{2.48}-2^{97}-1$$
$$=3.(3^2)^{48}-2^{97}-1$$
$$=3.9^{48}-2^{97}-1$$
$$\equiv 3.2^{48}-2^{97}-1 \pmod 7$$

And $3.2^{48}-2^{97}-1=(2+1).2^{48}-2^{97}-1=2^{49}+2^{48}-2^{97}-1$

So:

$$3^{97}-2^{97}-1\equiv 2^{49}+2^{48}-2^{97}-1 \pmod 7$$
$$\equiv 2^{49}+(2^3)^{16}-2^{97}-1 \pmod 7$$
$$\equiv 2^{49}+8^{16}-2^{97}-1 \pmod 7$$
$$\equiv (8^{16}-1)+2^{49}-2^{97} \pmod 7$$
$$\equiv 2^{49}-2^{97} \pmod 7$$
$$\equiv -(2^{97}-2^{49}) \pmod 7$$
$$\equiv -(2^{49}(2^{48}-1)) \pmod 7$$
$$\equiv -(2^{49}(8^{16}-1) \pmod 7$$
$$\equiv 0 \pmod 7 \quad (3)$$

From $(1), (2), (3)$ we have $(3^{97}-2^{97}-1) \vdots 42$ (**)

From (*) and ((**) we have: $(3^{97}-2^{97}-1) \vdots 42.97$

Exercise 8: Find all positive integers n such that $2^n-1 \vdots 7$

The answer:

As 2 a prime according to theorem little Fermat, we have $2^7 \equiv 2 \pmod 7$

From that $2^6 \equiv 1 \pmod 7 \Rightarrow 2^6-1\equiv 0 \pmod 7$, and $2^6-1=(2^3-1)(2^3+1)$, so $7 \mid (2^3-1)(2^3+1) \Rightarrow 7 \mid 2^3+1 \Leftrightarrow 2^3 \equiv 1 \pmod 7$

So all numbers n divisible by 3 satisfying requirement.

Exercise 9: Prove that there are infinitely many positive integers n that satisfy $(2^n-1) \vdots 67$

The answer:

As 2 a prime according to theorem little Fermat, we have $2^{67} \equiv 2 \pmod{67}$

From that $2^{66} \equiv 1 \pmod{67} \Rightarrow 2^{m.66} \equiv 1 \pmod{67}$, with m positive integer.

Take $n = m.66$, with $m \equiv -1 \pmod{67}$, we have $n = m.66 \equiv 1 \pmod{67}$

And $2^n - n \equiv 2^n - 1 \equiv 0 \pmod{67}$

So there are infinitely many positive integers m such that $m \equiv -1 \pmod{67}$ there are infinitely many positive integers n satisfying $(2^n - 1) \vdots 67$

Exercise 10: [5, tr. 45] Prove $3^{100} - 3 \vdots 13$

The answer:

As 13 a prime, according to theorem Fermat we have:

$3^{12} \equiv 1 \pmod{13}$

With $100 = 12.8 + 4$ so $3^{100} = (3^{12})^8.3^4 \equiv 3^4 \pmod{13}$

But $3^4 \equiv 81 \equiv 3 \pmod{13}$

Then the remainder in division $3^{100} - 3$ by 13 is 0, or $3^{100} - 3 \vdots 13$

Exercise 11: With $n \geq 2, a \geq 0$ is a positive integer and m is a prime such that $a^m \equiv 1 \pmod{m^n}$. Prove if $m > 2$ then $a \equiv 1 \pmod{m^{n-1}}$ and if $m = 2$ then $a \equiv \pm 1 \pmod{2^{n-1}}$

The answer:

We have $a^m \equiv 1 \pmod{m^n}$ with $n \geq 2$, so $a^m \equiv 1 \pmod{m}$

From theorem little Fermat $a^m \equiv a \pmod{m}$, so $a \equiv 1 \pmod{m}$

In case a = 1, Obviously there is something to prove

If $a \neq 1$, we put $a = 1 + km^u$ (here $u \geq 1$ and $k \not\vdots\, m$. So, $m > 2$, $a^m = 1 + km^{u+1} + Vm^{2u+1}$ with V an integer

Hence $m + 1 > n$ so $a \equiv 1 \pmod{m^{n-1}}$

In case $m = 2$, we have $2^n \mid a^2 - 1 = (a-1)(a+1)$

As $a - 1 \neq 2, a + 1 \neq 2$ so $a - 1, a + 1$ both cannot be multiples of 4. So either expression $a - 1, a + 1$ is divisible by 2^{n+1}, I.e $a \equiv \pm 1 \pmod{2^{n-1}}$. We have what to be proved

Exercise 12: With a as integer, knowing $a^{25} - a \vdots m \; (m > 1)$. Find *m*

The answer:

Suppose to find a number m that satisfies the requirements of the problem. Then with p is a prime number, because p^2 is not divisible by $p^{25} - p$, so we have p^2 not divisible by m. So m multiples of distinct primes

On the other hand, we have: $a^{25} - 2 = 2.3^2.5.7.13.17.241$

As $3^{25} \equiv -3 \pmod{17}$ and $3^{25} \equiv 32 \pmod{241}$ so m not divisible by 17 and 241.

According to theorem little Fermat, we have $a^{25} \equiv a \pmod{p}$ when $p = \{2; 3; 5; 7; 13\}$

Thus, m will be equal to the divisors of: 2. 3. 5. 7. 13, different from 1 and have $2^5 - 1 = 31$ their divisors.

Exercise 13: Let t be a positive integer, k be an odd natural number and $m = k.2^t + 1$ an odd prime number. Assume that for x, y are natural numbers satisfying $x^{2^t} + y^{2^t} \vdots m$. Prove that then x, y simultaneously divisible by m.

The answer:

We prove it by the method of contradiction

Assume the opposite, $x \not\vdots m$, then $y \not\vdots m$

As m a prime according to theorem little Fermat we have: $x^{m-1} \equiv 1 \pmod{m}, \ y^{m-1} \equiv 1 \pmod{m}$.

As $m - 1 = k.2^t$ then we have: $x^{k.2^t} \equiv 1 \pmod{m}, \ y^{k.2^t} \equiv 1 \pmod{m}$

From that we get $x^{k.2^t} + y^{k.2^t} \equiv 2 \pmod{m}$. (1)

According to assumption $x^{2^t} + y^{2^t} \vdots m \Leftrightarrow x^{2^t} + y^{2^t} \equiv 0 \pmod{m}$

Because k odd natural number:

$x^{k.2^t} + y^{k.2^t} = (x^{2^t})^k + (y^{2^t})^k \vdots (x^{2^t} + y^{2^t}) \equiv 0 \pmod{m}$ (2)

We see (1) and (2) contradict. So x and y at the same time divisible by m (what to be proved)

Exercise 14: Using theorem Fermat find remainder when divide 3456^{789} by 23

The answer:

According to theorem Fermat, we get:

$$3456^{789} \equiv (3456^{789})^{35} . 3456^{19}$$

$$\equiv 1^{35}.\,6^{19} \;(\bmod\; 23)$$
$$\equiv 6^{19} \;(\bmod\; 23)$$
$$\equiv 6^{18}.6 \;(\bmod\; 23)$$
$$\equiv 6^{3.6}.6 \;(\bmod\; 23)$$
$$\equiv (6^{3})^{6}.6 \;(\bmod\; 23)$$
$$\equiv 9^{6}.\,6 \;(\bmod\; 23)$$
$$\equiv (9^{2})^{3}.6 \;(\bmod\; 23)$$
$$\equiv 12^{3}.\,6 \;(\bmod\; 23)$$
$$\equiv 3.\,6 \;(\bmod\; 23)$$
$$\equiv 18 \;(\bmod\; 23)$$

So remainder when divide 3456^{789} by 23 is 18

Exercise 15: Prove $1^{18}+2^{18}+3^{18}+4^{18}+5^{18}+6^{18} \vdots 7$

The answer:

The numbers 1; 2; 3; 4; 5; 6 is a co-prime to 7, so by Fermat's theorem we have the following congruences:

$$1^{6} \equiv 1 \;(\bmod\; 7);$$
$$2^{6} \equiv 1 \;(\bmod\; 7);$$
$$3^{6} \equiv 1 \;(\bmod\; 7);$$
………………………….
$$6^{6} \equiv 1 \;(\bmod\; 7);$$

Raising to the third power on both sides of the above congruence, we get:

$$1^{18} \equiv 1 \;(\bmod\; 7)$$
$$2^{18} \equiv 1 \;(\bmod\; 7)$$
$$3^{18} \equiv 1 \;(\bmod\; 7)$$
………………………….
$$6^{18} \equiv 1 \;(\bmod\; 7)$$

Adding the sides of those congruent, we get:

$$1^{18}+2^{18}+3^{18}+4^{18}+5^{18}+6^{18} \equiv 0 \;(\bmod\; 7)$$
$$\Leftrightarrow 1^{18}+2^{18}+3^{18}+4^{18}+5^{18}+6^{18} \vdots 7$$

Exercise 16: With m is prime and $m>5$. Prove $m^8-1 \vdots 240$

The answer:

Analyze 240 Product analysis of prime factors (standard form), we get $240=2^4.3.5$

Apply little Fermat theorem, we have $m^4 \equiv 1 \pmod 5$ and $m^2 \equiv 1 \pmod 3$

Call a positive integer, we see $(a, 2^4)=1$ when a is odd.

On the other hand apply formula $\varphi(m)=m(1-\frac{1}{p_1})(1-\frac{1}{p_2})....(1-\frac{1}{p_k})$ is the number of smaller than m and prime positive numbers with m (với $m=p_1^{m_1}.p_2^{m_2}....p_k^{m_k}$ is the standard analysis of m), we get $\varphi(24)=\varphi(2^3.3)=24.(1-\frac{1}{2}).(1-\frac{1}{3})=8=2^3$

According to theorem Euler, we get $m^8 \equiv 1 \pmod{16}$.

Hence $m^8 \equiv 1 \pmod p$ with $p=3, 5, 16$. From that $m^8-1 \vdots 240$ (what to be proved)

Exercise 17: With p as a prime and $p-2 \vdots 3$. Prove with a,b are integers satisfying $a^3-b^3 \vdots p$ then $a-b \vdots p$

The answer:

Consider 2 cases: a or b divisible by p and a,b are not divisible by p

+) Case 1: If $a \vdots p \Rightarrow b^3 \vdots p \Rightarrow b \vdots p$, so $a-b \vdots p$

+) Case 2: If a,b are not divisible by p

As assumption $p-2 \vdots 3 \Rightarrow p=3k+2$ $(k \in \mathbb{})$

According to little Fermat theorem, we get $a^{p-1} \equiv 1 \pmod p$

But $p=3k+2$ $(k \in \mathbb{})$ then $a^{3k+1} \equiv 1 \pmod p$

Similar proof, we get $b^{3k+1} \equiv 1 \pmod p$

From that $a^{3k+1}-b^{3k+1} \vdots p$

According to assumption $a^3-b^3 \vdots p \Rightarrow a^{3k}-b^{3k} \vdots p$

From that $a-b \vdots p$ (what to be proved)

Conclusion

The diverse and complex topics of Arithmetic problems are always attractive to good students, because solving each problem in this field helps train and develop students' thinking. Using Fermat's little theorem to solve divisibility problems, in addition to providing quick, concise and accurate solutions, also stimulates new discoveries and discoveries in problem solving, helping learners to be creative in learning. Through this type of math, learners have the ability to think deeply, the ability to flexibly apply knowledge content to each type of exercise accordingly, so that they can achieve the best effect to meet the program. new general education with the goal of orienting student capacity development.

Acknowledgement

This research is funded by Tan Trao University in Tuyen Quang, Viet Nam.

References

Arthur Engel, Problem-Solving Strategies, Springer, 1998.

Bui Trong Kien, Theoretical and technical basis for solving Olympic math problems, Hanoi National University Publishing House, 2017.

DT Tinh et al. (2021). Doing Business Research and Teaching Methodology for Undergraduate, Postgraduate and Doctoral Students-Case in Various Markets Including Vietnam, *Elementary education Online*, 20(1).

Do Thu Huong et al. (2021). Discussion on Case Teaching Method in a Risk Management Case Study with Econometric Model at Vietnam Listed Banks–Issues of Economic Education for Students. *Review of International Geographical Education Online* 11(5), 2957-2966.

Nguyen Vu Luong (editor), Nguyen Luu Son, Nguyen Ngoc Thang, Pham Van Hung, Lectures on Arithmetic, (volume one), Hanoi National University Press, 2006.

F Yong-Yan, J Manafian, SM Zia, DTN Huy, TH Le. (2021). Analytical Treatment of the Generalized Hirota-Satsuma-Ito Equation Arising in Shallow Water Wave, Advances in Mathematical Physics 2021.

G Shen, J Manafian, SM Zia, DTN Huy, TH Le. (2021). The New Solitary Solutions to the Time-Fractional Coupled Jaulent–Miodek Equation, Discrete Dynamics in Nature and Society 2021.

Ha Huy Khoai, Arithmetic, Hanoi Education Publishing House, 2004.

Guiping ShenaJalilManafian, Dinh Tran Ngoc Huy, Kottakkaran Sooppy, Nisare, Mostafa Abotale, Nguyen DinhTrung. (2021). Abundant soliton wave solutions and the linear superposition principle for generalized (3+1) nonlinear wave equation in liquid with gas bubbles by bilinear analysis, Results in Physics, 32. https://doi.org/10.1016/j.rinp.2021.105066

Le Thi Thanh Huong et al. (2021). Education for students to enhance research skills and meet demand from workplace - case in vietnam. *Elementary education online*, 20(4): 606-611.

Nguyen Vu Luong (editor), Nguyen Luu Son, Nguyen Ngoc Thang, Pham Van Hung, Lectures on Arithmetic, (volume two), Hanoi National University Press, 2006.

Hoang Chung, Arithmetic, Hanoi Education Publishing House, 1997.

Ha Huy Khoai, Pham Huy Diem, Algorithmic Arithmetic and Practical Computation, Hanoi National University Press, 2003.

Nguyen Dinh Thanh Cong, Nguyen Van Huong, Nguyen Duy Hung, Tran Chi Kien, Nguyen Van Son, Le Nhat, Tran Bao Trung, Topics on fostering good students through Math Olympiad competitions, National University Press Hanoi, 2016.

Nguyen Thi Hang et al. (2021). Educating and training labor force Under Covid 19 Impacts to Meet Market Demand in Vietnam during Globalization and Integration Era, ESCI. *Journal for Educators, Teachers and Trainers*, Vol.12(1).

Tran Dien Hien, Nguyen Tien Tai, Nguyen Van Ngoc, Textbook of number theory, Education Publishing House, Hanoi, 1997.

Duong Quoc Viet, Dam Van Nhat, Basis of Number Theory and Polynomials, Hanoi National University Press, 2003.

Pham Minh Phuong, Some complex math topics on fostering excellent high school students, Vietnam Education Publishing House, Hanoi, 2010.

PN Tram et al. (2021). Educational, Political and Socio-Economic Development of Vietnam Based on Ho Chi Minh's Ideology. *Elementary education Online* 20(1).

Le Hai Chau, National High School Math Good Students Exams, Vietnam Education Publishing House, Hanoi, 1994.

Junior High School Childhood Math Journal No. 168 + 169, Vietnam Education Publishing House, Hanoi, 2017.

Titu Andreescu, Razvan Gelca, Mathematical Olypiad Challenges, Bikhauser, 2000.

TDT Vu et al. (2021). Human Education and Educational Issues for Society and Economy-Case In Emerging Markets Including Vietnam. *Elementary education Online*, 20 (2)

Van Phu Quoc, Breakthrough at the pinnacle of fostering excellent students in arithmetic, Hanoi National University Publishing House, 2015.

Youth Mathematics Journal, Vietnam Education Publishing House, Hanoi, 2011.