Turkish Online Journal of Qualitative Inquiry (TOJQI)
Volume 12, Issue 10, October 2021: 5237-5245

The Minimal Connected Geo Chromatic Number of a Graph

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Abstract:

For a connected graph G of order n ≥ 2, a connected geo chromatic set S_{cg} in a connected graph G is called a minimal connected geo chromatic set if no proper subset of S_{cg} is a connected geo chromatic set of G. The minimal connected geo chromatic number $\chi_{cg}^{+}(G)$ is the maximum cardinality of a minimum connected geo chromatic set of G. We determined the minimum connected geo chromatic number of certain standard graphs and bounds of the minimum connected geo chromatic number is proved. It is shown that for positive integers x, y and z such that $2 \leq x < y \leq z$, there exists a connected graph G such that g(G) = x, $\chi_{cg}(G) = y$ and $\chi_{cg}^{+}(G) = z$.

Keywords : geodetic number, chromatic number, geo chromatic number, connected geo chromatic number, minimum connected geo chromatic number.

2000 Mathematics Subject Classification : 05C12, 05C15.

1 INTRODUCTION

Let $G = (V, E)$ be a finite undirected connected graph without multiple edges or loops. The order and size of G are denoted by n and m respectively. For basic graph theoretic terminology we refer to Harary [7]. For vertices p and q in a connected graph G, the distance $d(p, q)$ is the length of a shortest $p - q$ path in G. A $p - q$ path of length $d(p, q)$ is called a $p-q$ geodesic. A vertex x is said to lie on a $p-q$ geodesic p', if x is a vertex of p', including the vertices of p and q. The neighborhood of a vertex x is the set $N(x)$ consisting of all vertices y which are adjacent with x. A vertex x is an extreme vertex of G if the subgraph induced by its neighbors is complete.

The closed interval $I[p, q]$ consists of all vertices lying on some $p - q$ geodesic of G, while for $S \subseteq V$, $I[S] = \cup_{p,q \in S} I[p, q]$. If $I[S] = V$, then a set S of vertices is a geodetic set and the minimum cardinality of a geodetic set is the geodetic number $g(G)$. A geodetic number of a graph was introduced in [3,4] and further studied in [6,8].

A connected geodetic set of G is a geodetic set S' such that the subgraph $G[S']$ induced by S' is connected. The minimum cardinality of a connected geodetic set of G is the connected geodetic number of *G* and is denoted by $g_c(G)$. The concept of the connected geodetic number was studied in [9,11].

The concept of geo chromatic number was introduced by S. B. Samli and S. R. Chellathurai in [1] and further studied in [2,10]. A geodetic set *S* is said to be a geo chromatic set S_c of *G*, if *S* is both a geodetic set and a chromatic set of *G*. The minimum cardinality of a geo chromatic set of *G* is the geo chromatic number of *G* and is denoted by $\chi_{gc}(G)$.

The concept of connected geo chromatic number was introduced by S. Elizabeth Bernie, M. Ashwin Shijo and S. B. Samli [5]. A geo chromatic set S_c is said to be a connected geo chromatic set if the subgraph $<S_c>$ induced by S_c is connected. The minimum cardinality of a connected geo chromatic number of *G* is the connected geo chromatic number and is denoted by $\chi_{cg}(G)$. A connected geo chromatic set of cardinality $\chi_{cg}(G)$ is called a χ_{cg}- set of *G*.

In this paper we introduce the new concept as minimal connected geo chromatic number of a graph. In section 2, we introduce the definition of minimal connected geo chromatic number, we determine the minimal connected geo chromatic number of some standard graphs and general results. In section 3, we illustrate realization of the minimal connected geo chromatic number of *G*. The following theorems used in sequel.

Theorem 1.1. [8] For any tree *T* with *k* end vertices, $g(T) = k$.

Theorem 1.2. [4] Every extreme vertex of a connected graph *G* belongs to every connected geodetic set of *G*.

Theorem 1.3. [5] Every cut vertex of a connected graph *G* belongs to every connected geo chromatic set of *G*.

Theorem 1.4. [5] For any tree *T* with *n* vertices, $\chi_{cg}(T) = n$.

Theorem 1.5. [5] For any connected graph *G*, $\chi_{cg}(G) \geq diam\,(G) + 1$.

2 DEFINITIONS AND EXAMPLES

Definition 2.1. A connected geo chromatic set S_{cg} in a connected graph *G* is called a minimal connected geo chromatic set if no proper subset of S_{cg} is a connected geo chromatic set of *G*. The minimal connected geo chromatic number $\chi^+_{cg}(G)$ is the maximum cardinality of a minimum connected geo chromatic set of *G*.

Example 2.2. For the graph *G* given in Figure 1, $S_{cg1}= \{a_2,a_4,a_5, a_6\}$, $S_{cg2}= \{a_1,a_2, a_4, a_5\}$, $S_{cg3}= \{a_1,a_2, a_4, a_6\}$, $S_{cg4}= \{a_2,a_3, a_4, a_5\}$, $S_{cg5}=\{a_2,a_3, a_4, a_6\}$ are the minimum connected geo chromatic set of *G* so that $\chi_{cg}(G)=4$. The set $S^+_{cg}= \{a_1,a_3,a_4,a_5,a_6\}$ is also a connected

geo chromatic set of *G* . Hence $\chi^+_{cg}(G) = 5$.

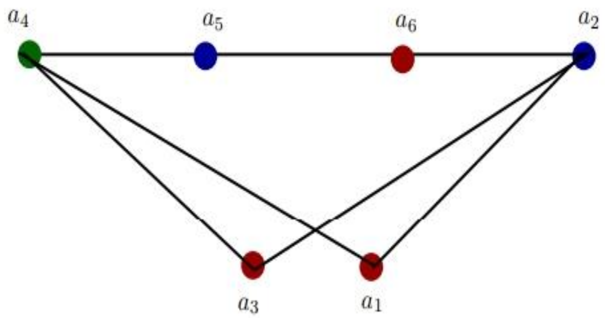

Figure 2 : G

Remark 2.3. Every minimum connected geo chromatic set of G is a minimal connected geo chromatic set of G. The converse is not true. For the graph G given Figure 1, S^{+}_{cg}= $\{a_1,a_3,a_4,a_5,a_6\}$ is a minimal connected geo chromatic set but not a minimum connected.

Theorem 2.4. For the cycle C_n, $\chi^{+}_{cg}(C_n) = \begin{cases} \frac{n}{2}+1 & if\ \ n \text{ is even} \\ \left\lfloor \frac{n}{2}+2 \right\rfloor & if\ n \text{ is odd} \end{cases}$.

Proof. **Case : 1** Suppose that *n* is even.

Let *n* = 2*p*. Let C_{2p}: a_1,a_2,…,a_{2p}, a_1 be the cycle of order 2*p*. Let

S'_{cg}=$\{a_1,a_2,\dots,a_{p+1}\}$. It is clear that S'_{cg} is a connected geo chromatic set of C_n. We prove that S'_{cg} is a minimal connected geo chromatic set of C_n. Let $S'_{cg} \subset S''_{cg}$. Then there is a vertex say *v* $\in S'_{cg}$ such that $v \notin S''_{cg}$. If *v* = a_1 or a_{p+1}, then S''_{cg} is not a geo chromatic set of C_n. If v $\neq a_1$ and v $\neq a_{p+1}$, then the subgraph induced by S''_{cg} is not connected. Hence it follows that S'_{cg} is a minimal connected geo chromatic set of C_n and so $\chi^{+}_{cg}(C_n) \geq |S'_{cg}|$=*p* +1.

Now, we claim that $\chi^{+}_{cg}(C_n)$ = *p* + 1. Otherwise, there is a minimal connected geo chromatic set *A* such that |*A*| = l > *p* + 1. Since *A* is a connected geo chromatic set of C_n, the subgraph induced by *A* is a path, say a_{j+1},a_{j+2},…,a_{j+m}. It is clear that *W*=$\{a_{j+1},a_{j+2},\dots,a_{j+p+1}\}$ is a connected geo chromatic set of C_n and so *A* is not a minimal connected geo chromatic set of

C_n, which is a contradiction. Thus $\chi^{+}_{cg}(C_n)$ =*p*+1.

Case : 2 Suppose that *n* is even.

Let *n* = 2*p* + 1. Let C_{2p+1}: a_1,a_2,…,a_{2p+1}, a_1 be the cycle of order 2*p*+1. Let

S'_{cg}=$\{a_1,a_2,\dots,a_{p+1}, a_{p+2}\}$. Then, as in Case 1, it is seen that S'_{cg} is a minimal connected geo chromatic set of C_n and $\chi^{+}_{cg}(C_n)$ =*p* + 2.

Theorem 2.5. Every extreme vertex of a connected graph *G* belongs to every minimal connected geo chromatic set of *G*.

Proof. Since every minimal connected geo chromatic set is a geo chromatic set, the result follows from Theorem 1.2.

Theorem 2.6. Every minimal connected geo chromatic set of G contains every cut vertex of a connected graph G.

Proof. Let c' be any cut vertex of G and let $A_1,A_2,\ldots,A_r$ $(r \geq 2)$ be the components of $G-\{c\}$. Let S^+_{cg} be any minimal connected geo chromatic set of G. First we claim that S^+_{cg} contains at least one element from every component A_i $(1 \leq i \leq r)$. Suppose that A_1contains no vertex of S^+_{cg}. Let $d \in V$ (A_1) . Since S^+_{cg} is a minimal connected geo chromatic set, there exists a pair of vertices k and l in S^+_{cg} such that d lies in some $k - l$ geodetic

Q': $k = d_0,d_1,\ldots,d,\ldots,d_n= l$ in G. Since c' is a cut vertex of G, the $k - d$ subpath of Q' and the $d-l$ subpath of Q' both contain c', it follows that Q' is not a path, which is a contradiction. Thus S^+_{cg} contains at least one element from each A_i $(1 \leq i \leq r)$. Since $G[S^+_{cg}]$ is connected, it follows that $c' \in S^+_{cg}$.

Corollary 2.7. For a connected graph G with e' extreme vertices and c' cut vertices, $\chi^+_{cg}(G) \geq \max\ \{2, e' + c'\}$.

Proof. This follows from Theorems 2.5 and 2.6.

Corollary 2.8. For the complete graph K_n, $\chi^+_{cg}(K_n) = n$.

Proof. This follows from Theorem 2.5.

Corollary 2.9. For any tree T, $\chi^+_{cg}(T) = n$.

Proof. This follows from Corollary 2.7.

Theorem 2.10. For the complete bipartite graph $G = K_{p,q}$

1. $\chi^+_{cg}(G) = 2$ if $p = q = 1$.
2. $\chi^+_{cg}(G) = q + 1$ if $p = 1, q \geq 2$.
3. $\chi^+_{cg}(G) = \max\ \{p, q\} + 1$ if $p, q \geq 2$.

Proof. (1) and (2) follows from Corollary 2.9.

(3) Without loss of generality, let $p \leq q$. Let $A'= \{a_1,a_2,\ldots,a_p\}$ and $B'= \{b_1,b_2,\ldots,b_q\}$ be a bipartition of G. Let $S^+_{cg}= B' \cup \{a\}$, where $a \in A'$. We prove that S^+_{cg} is a minimal connected geo chromatic set of G. It is clear that each vertex of A' lies on a geodesic joining a pair of vertices of B'. Also, since $G[S^+_{cg}]$ is connected, S^+_{cg} is a connected geo

chromatic set of G. Let $S'_{cg} \subsetneq S^+_{cg}$. If a$\notin S'_{cg}$, then $S'_{cg}$is not a connected geo chromatic set of G. If $a \in S'_{cg}$, then since $|S'_{cg}| < |S^+_{cg}|$, there exists at least one $b_i \in S^+_{cg}$ such that $b_i \notin S'_{cg}$for some i ($1 \leq i \leq q$) . It is clear that b_i does not lie on any geodesic joining any pair of vertices of S'_{cg}. Thus S'_{cg} is not a geo chromatic set of G and so S^+_{cg} is a minimal connected geo chromatic set of G. Hence $\chi^+_{cg}(G)= q + 1$. Let

A'' be any connected geo chromatic set of G such that $|A''| \geq q + 2$. Since $p \leq 2$, A'' contains at least two elements from A'', say a_i and a_j and at least two elements from B', say b_k and b_l. It is clear that $A^+= \{a_i, a_j, b_k, b_l\}$ is a connected geo chromatic set of G and so A'' is not a minimal connected geo chromatic set of G. Thus $\chi^+_{cg}(G) = q +1$.

Theorem 2.11. For a connected graph G, $2 \leq \chi_{cg}(G) \leq \chi^+_{cg}(G) \leq n$.

Proof. Any connected geo chromatic set needs at least two vertices and so $\chi_{cg}(G) \geq 2$. Since every minimum connected geo chromatic set is a minimal connected geo chromatic set, $\chi_{cg}(G) \leq \chi^+_{cg}(G)$. Also, since $V(G)$ induces a connected geo chromatic set of G, it is clear that $\chi^+_{cg}(G) \leq n$. Thus $2 \leq \chi_{cg}(G) \leq \chi^+_{cg}(G) \leq n$.

Remark 2.12. For the graph K_2, $\chi^+_{cg}(K_2)= 2$. For any non-trivial tree T of order n, $\chi^+_{cg}(K_2) = 2$. Also, all the inequalities in Theorem 2.11 are strict. For the graph G given in Figure 1, $\chi_{cg}(G) = 4$, $\chi^+_{cg}(G) =5$ and $n = 6$ so that $2 < \chi_{cg}(G) < \chi^+_{cg}(G) < n$.

Theorem 2.13. For a connected graph G, $\chi_{cg}(G) = n$ if and only if $\chi^+_{cg}(G) = n$.

Proof. Let $\chi^+_{cg}(G) = n$. Then $S^+_{cg}= V(G)$ is the unique minimal connected geo chromatic set of G. Since no proper subset of $\chi^+_{cg}(G)$ is a connected geo chromatic set, it is clear that $\chi^+_{cg}(G)$ is the unique minimum connected geo chromatic set of G. The converse follows from Theorem 1.5 and 2.11.

Theorem 2.14. For the sunlet graph S_n ($n \geq 3$), $\chi^+_{cg}(S_n) = 2n$.

Proof. Let $V(S_n) = \{a_1,a_2,\ldots,a_n,a'_1,a'_2,\ldots,a'_n\}$ be the vertices of the cycle C_n and the set $\{a'_1,a'_2,\ldots,a'_n\}$ be the pendant vertices which is adjacent to C_n. It is clear that the vertex set $\{a'_1,a'_2,\ldots,a'_n\}$ is a minimum geodetic set S of S_n and so $g(S_n)=n$. Also, it is easily seen that $S =\{a'_1,a'_2,\ldots,a'_n\}$is a minimum chromatic set C of S_n. Therefore the vertex set $\{a'_1, a'_2, \ldots, a'_n\}$ is the unique minimum geo chromatic set of S_n and $\chi_{gc}(S_n) = n$. But the induced subgraph < S_c> is not connected, so that S_c is not a connected geo chromatic set of S_n. Let $S'_c = \{a_2, a_3, \ldots, a_n, a'_1, a'_2, \ldots, a'_n\}$. Clearly the induced subgraph < S'_c> is not connected and so $\chi_{cg}(S_n) \geq 2n$. The only possible connected geo chromatic set is $S''_c = V(S_n)$, and so $\chi_{cg}(S_n) = 2n$. Since $V(S_n) = 2n$ and $\chi_{cg}(G) \leq \chi^{+}_{cg}G)$, $\chi^{+}_{cg}(S_n) = 2n$.

3 REALIZATION RESULTS

Theorem 3.1. For any positive integers $2 \leq x < y \leq z$, there exists a connected graph G such that $g(G) = x$, $\chi_{cg}(G) = y$ and $\chi^{+}_{cg}(G) = z$.

Proof. If $2 \leq x < y = z$, let G be any tree of order y with x end vertices. Then by Theorem 1.1, $g(G) = x$, by Theorem 1.4, $\chi_{cg}(G) = y$ and by Corollary 2.9, $\chi^{+}_{cg}(G) = z$. Let $2 \leq x < y < z$.

Now, we consider the following four cases.

Case : 1 Let $x > 2$ and $y - x \geq 2$.

Then $y - x + 2 \geq 4$. Let P_{y-x+2}: $a_1, a_2, \ldots, a_{y-x+2}$ be a path of length $y - x + 1$. Add z- y+ x-1 new vertices $b_1, b_2, \ldots, b_{z-y+1}$, $c_1, c_2, \ldots, c_{x-2}$ to $P_{b-a+2}$and join $b_1, b_2, \ldots, b_{z-y+1}$ to both a_1 and a_3 and also join $c_1, c_2, \ldots, c_{x-2}$ to a_2, there by producing the graph G of Figure 2.

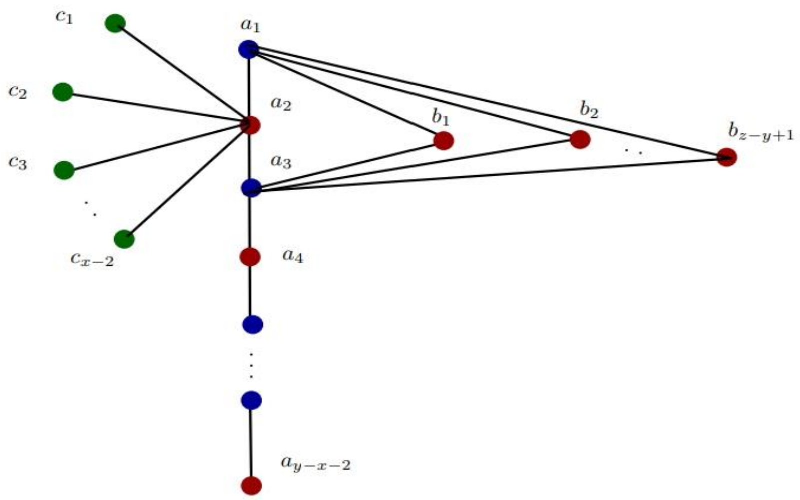

Figure 2: G

Let the set of all extreme vertices of G be $E = \{c_1, c_2, \ldots, c_{x-2}, a_{y-x+2}\}$. By Theorem 1.2, every geodetic set S of G contains E. It is clear that E is not a geodetic set S of G. But $E \cup \{a_1\}$ is a geodetic set S of G so that $g\ (G) = x$. Let $S' = E \cup \{a_2, a_3, \ldots, a_{y-x+2}\}$. By Theorem 1.2, and Theorem 1.3, every connected geo chromatic set contains S'. It is clear that S' is not a connected geo chromatic set of G. But $S' \cup \{a_1\}$ is a connected geo chromatic set of G so that $\chi_{cg}(G) = y$.

Let $S'' = S' \cup \{b_1, b_2, \ldots, b_{z-y+1}\}$. It is clear that S'' is a connected geo chromatic set of G. Now, we show that S'' is a minimal connected geo chromatic set of G. Assume to the contrary that S'' is not a minimal connected geo chromatic set. Then there is a proper subset F of S'' such that F is a connected geo chromatic set of G. Let $a \notin F$. By Theorem 1.2 and 1.3, it is clear that $a = b_i$ for some $i = 1, 2, \ldots, z - y + 1$. Clearly, this b_i does not lie on a geodesic joining any pair of vertices of F and so F is not a connected geo chromatic set of G, which is a contradiction. Thus S'' is a minimal connected geo chromatic set of G and so $\chi^+_{cg}(G) \geq z$. Since the order of the graph is $z + 1$, it follows that $\chi^+_{cg}(G) = z$.

Case : 2 Let $x > 2$ and $y - x = 1$.

Then $y = 3$. Consider the graph G given in Figure 3. Then, as in Case 1, $S = \{c_1, c_2, \ldots, c_{x-2}, a_1, a_3\}$ is a minimum geodetic set of G, $S_{cg} = S \cup \{a_2\}$ is a minimum connected geo chromatic set of G and $S^+_{cg} = V(G) - \{a_1\}$ is a minimal connected geo chromatic set of G so that $g(G) = x$, $\chi_{cg}(G) = y$ and $\chi^+_{cg}(G) = z$.

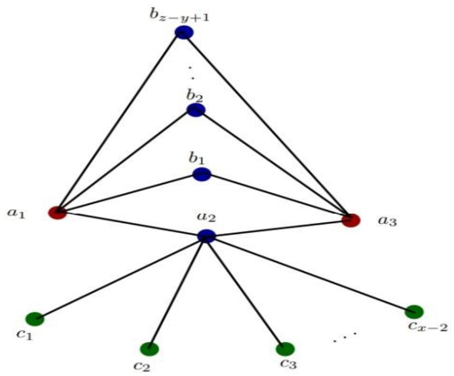

Figure 3: G

Case : 3 Let $x = 2$ and $y - x = 1$.

Then $y = 3$. Consider the graph G given in Figure 4. Then, as in Case 1, $S = \{a_1, a_3\}$ is a minimum geodetic set of G, $S_{cg} = \{a_1, a_2, a_3\}$ is a minimum connected geo chromatic set of G and $S^+_{cg} = V(G) - \{a_1\}$ is a minimal connected geo chromatic set of G so that $g(G) = x$, $\chi_{cg}(G) = y$ and $\chi^+_{cg}(G) = z$.

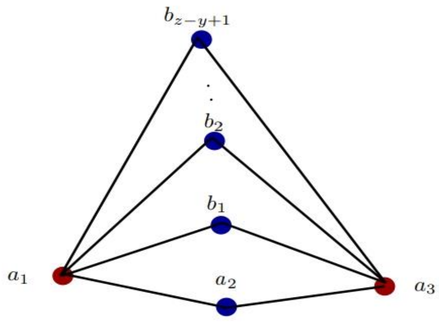

Figure 4: G

Case : 4 Let $x = 2$ and $y - x \geq 2$.

Then $y \geq 4$. Consider the graph G given in Figure 5. Then, as in Case 1, $S = \{a_1, a_y\}$ is a minimum geodetic set of G, $S_{cg} = \{a_1, a_2, a_y\}$ is a minimum connected geo chromatic set of G and $S^+_{cg} = V(G) = \{a_1\}$ is a minimal connected geo chromatic set of G so that $g(G) = x$, $\chi_{cg}(G) = y$ and $\chi^+_{cg}(G) = z$.

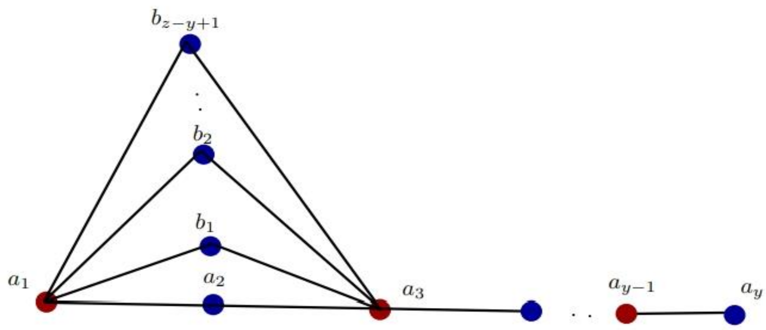

Figure 5: G

4 CONCLUSION

In this paper, the minimal connected geo chromatic number $\chi^+_{cg}(G)$ has been discussed. We have presented bounds and general results of the minimal connected geo chromatic number. Also, the realization results involving the minimal connected geo chromatic number were discussed.

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