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An Introduction to Laplace Transforms

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Abstract:
In this paper we shall introducd the Laplce transforms,develop its fundamental properties and illustrate its use to solve linear differential equations with bounday conditions. The paper gives theory, few examples worked on properties and application of Laplace transform.
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1.Introduction:

The laplce transform is a powerful tool for solving linear differerntial equations with given boundary values.Normally to find the solution of an equation with given boundary conditions,we first find the genearl solutions of the equation,the evaluate the value of the arbitrary constants.

2. Laplace Transform definition:

Consider a function f(t) of real variable t defined for t≥0.Laplace transform of f(t) denoted by L[f(t)] ,given by

$L[f(t)]=\int_0^\infty e^{-st} f(t)dt$

Provided the integral on right hand side exists where s is a parameter, real or complex number.
Thus

$L[f(t)]=\int_0^\infty e^{-st} f(t)dt = F(s)$

If F(s) is the Laplace transforms of f(t),then f(t) is called the inverse Laplace transform given by
$f(t)=L^{-1}[F(s)]$

Example:

$L[f(t)] = e^t$

$L[f(t)]= L[e^t] = \int_0^\infty e^{-st} e^t\, dt = \int_0^\infty e^{(1-s)t}\, dt$

$= \frac{1}{1-s} e^{(1-s)t} \Big|_0^\infty$

$= \frac{1}{s-1}$

3. Properties of Laplace transform:

Linearity property

If f(t) and g(t) are two functions whose Laplace transforms exists, and a & b are constants then L[af(t)+bg(t)]=aL[f(t)]+bL[g(t)]

First shifting property

If Laplace transform of f(t) is F(s) then the Laplace transform of $e^{at}f(t) = F(s-a)$

i.e. $L[e^{at}f(t)] = F(s-a)$

By definition, L[f(t)]=$\int_0^\infty e^{-st}$f(t)dt

$L[e^{at}f(t)$ =$\int_0^\infty e^{-st}\ \ e^{at}$f(t)dt

=$\int_0^\infty e^{-(s-a)t}$f(t)dt

=$F(s-a)$

Change of Scale Property

L[f(at)]=$\frac{1}{a}$F($\frac{s}{a}$)

Laplace transform of t^n where n is a positive integer

L[t^n]=$\frac{n!}{s^{n+1}}$

Example:[$3t^2$]=3L[t^2] =$3\left[\frac{2!}{s^{2+1}}\right] = 3\left[\frac{2}{s^2}\right] = \frac{6}{s^2}$

Laplace transform of multiplication of power of a variable.

The Laplace transform can be expressed as

L[t^n f(t)]=$(-1)^n$ F^n(s)

By definition, L[f(t)]=$\int_0^\infty e^{-st}$f(t)dt

F'(s)=$\frac{d}{ds}\int_0^\infty e^{-st}$f(t)dt

=$\int_0^\infty \frac{d}{ds} e^{-st}$f(t)dt

=$\int_0^\infty -te^{-st}$f(t)dt

F′(s)=$-\int_0^\infty e^{-st}(t$f(t))dt=-L[tf(t)]

F′′(s)= - $\int_0^\infty (-t)e^{-st}$ [tf(t)]dt

=$(-1)^2 \int_0^\infty e^{-st}$ [t^2f(t)]dt

=$(-1)^2$ L[t^2f(t)]

In general, we have

L[t^n f(t)]=$(-1)^n$ F^n(s)

Example: $L[t^2\cos 2t] = (-1)^2\frac{d^2}{ds^2}\left[\frac{s}{s^2+4}\right]$

$= \frac{d}{ds}\left[\frac{1(s^2+4)-2s(s)}{(s^2+4)^2}\right. = \frac{d}{ds}\left[\frac{-s^2+4}{(s^2+4)^2}\right]$

$= \frac{-2s(s^2+4)^2-(-s^2+4)2(s^2+4)(2s)}{(s^2+4)^4}$

$= \frac{(s^2+4)(2s)[-(s^2+4)-2(-s^2+4)]}{(s^2+4)^4} = \frac{(2s)[(s^2-4)]}{(s^2+4)^3}$

Laplace transform of division of variables

$L[\frac{1}{t}f(t)] = \int_s^\infty F(s)ds$

By definition, $L[f(t)] = \int_0^\infty e^{-st}f(t)dt$

$\int_0^\infty F(s)ds = \int_0^\infty [\int_0^\infty e^{-st}f(t)dt]ds$

$\int_s^\infty F(s)ds = \int_0^\infty [\int_s^\infty e^{-st}f(t)ds]dt = \int_0^\infty [\frac{e^{-st}}{-t}\ l_0^\infty f(t)dt$

$= \int_0^\infty e^{-st}[\frac{f(t)}{t}]dt = L[\frac{1}{t}f(t)]$

$\boldsymbol{\int_s^\infty F(s)ds = L[\frac{1}{t}f(t)]}$

Example: $L[\frac{1}{t}\sin t] = \int_s^\infty \frac{1}{1+s^2}ds$

$= \tan^{-1}s \Big|_s^\infty$

$= \tan^{-1}(\infty) - \tan^{-1}(s) = \frac{\pi}{2} - \tan^{-1}(s)$

Laplace Transform of Derivatives

Let f (t) be the time domain function. The Laplace Transform of derivative is given by

$L[f'(t)] = s\,L[f(t)] - f(0)$

By definition,

$L[f'(t)] = \int_0^\infty e^{-st}f'(t)dt$

Integrating by parts,

$L[f'(t)] = e^{-st}f'(t)I_0^\infty - \int_0^\infty (-s)e^{-st}f'(t)dt$

$= f(0) + s\int_0^\infty e^{-st}f'(t)dt$

$L[f'(t)] = sL[f(t)] - f(0)$

$L[f''(t)] = s^2L[f(t)] - sf(0) - f'(0)$

In general, $L[f^n(t)] = s^nL[f(t)] - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-2}f''(0) - \ldots f(o)$

3.8 Laplace transform of integrals:

$L[\int_0^t f(u)du] = \frac{1}{t}L[f(t)]$

D. Inverse Laplace transform:

If F(s) is the Laplace transforms of f(t),then f(t) is called the inverse Laplace transform given by $f(t) = L^{-1}[F(s)]$

E. Convolution theorem:

If f(t) and g(t) are two functions, then the integral

$\int_0^t f(u)g(t-u)du$ is called the convolution of f(t) and g(t) and is denoted by f(t)*g(t)

Thus f(t)*g(t)= $\int_0^t f(u)g(t-u)du$

Theorem:

If L[f(t)]=F(s) and L[g(t)]=G(s)

Then L-1[F(s) G(s)]= $\int_0^t f(u)g(t-u)du$

F. Laplace transform of periodic functions:

The Laplace transform of periodic function f(t) with period T is given by

L[f(t)]=$\frac{1}{1-e^{sT}}\int_s^T e^{-st}$f(t)dt

By definition,

L[f(t)]=$\int_0^\infty e^{-st}$f(t)dt

=$\int_0^T e^{-st}$f(t)dt+$\int_T^\infty e^{-st}$f(t)dt

In the second integral put t=u+T then dt=du

L[f(t)] =$\int_0^T e^{-st}$f(t)dt+$\int_T^\infty e^{-s(u+T)}$f(u+T)du

= $\int_0^T e^{-st}$f(t)dt+ $\int_T^\infty e^{-su}$f(u)du

L[f(t)]=$\int_0^T e^{-st}$f(t)dt+e^{-st}L[(f(t)]

(1-e^{-st})L[(f(t)]= $\int_0^T e^{-st}$f(t)dt

L[(f(t)]= $\frac{1}{(1-e^{-st})}\int_0^T e^{-st}$f(t)dt

Application of Laplace transforms to solve differential equations:

1.Solve $(D^4 - K^4)y = 0$, Using Laplace transforms, where y(0) = 1, y'(0) = y''(0) = y'''(0) = 0

$L(D^4y) - L(K^4y) = 0$,

Taking Laplace transform of each individual term of the given equation,

$L(D^4y) - K^4 L(y) = 0$

$S^4F(s)-s^3 y(0)-s^2 y'(0)-s y''(0)-y'''(0)-K^4 F(s)=0$

$S^4F(s)-s^3 (1) -s^2 (0)- s(0)-(0)-K^4 F(s)=0$

$S^4F(s)- K^4 F(s)= s^3$

$(S^4-K^4)F(s)= s^3$

F(s)= $\frac{s^3}{s^4-k^4} = \frac{s^3}{(s+k)(s-k)(s^4+k^4)}$

$\frac{s^3}{(s+k)(s-k)(s^4+k^4)}=\frac{A}{s-K}+\frac{B}{s+K}+\frac{Cs+D}{s^2+k^2}$

$s^3 = A(s+k)(s^2+k^2) + B(s-k)(s^2+k^2) + (Cs+D)(s^2+k^2)$

After solving we get A= $\frac{1}{4}$ B= $\frac{1}{4}$ C=0 D=$\frac{1}{2}$

$\frac{s^3}{(s+k)(s-k)(s^4+k^4)} = \frac{1}{4(s-K)} + \frac{1}{4(s+K)} + \frac{1}{2(s^2+k^2)}$

Taking inverse Laplace transform of each term we get

$Y(t) = \frac{1}{4}e^{kt} + \frac{1}{4}e^{-kt} + \frac{1}{2}\cos Kt$

$= \frac{1}{4}\left(e^{kt} + e^{-kt}\right) + \frac{1}{2}\cos Kt = \frac{1}{2}\cosh kt + \frac{1}{2}\cos Kt = \frac{1}{2}(\cosh Kt + \cos kt)$

Laplace transforms are used in solving Circuit analysis

The Laplace transform are used in analysing electrical circuits . By using Laplace transforms we can analyse an electrical circuit to discover its current, its maximum capacity. This is important for engineers, electrical engineers, in doing their jobs to ensure the necessary machines and technology is working properly.

Conclusion:

The transform has many applications in science and engineering because it is a tool for solving differential equations. It transforms linear differential equations into algebraic equations and convolution into multiplication. Laplace transforms have come an integral part of the current world. It is used in a vast number of different disciplines. Laplace transforms are used in electrical circuit analysis, signal processing, or in modelling radioactive decay in nuclear drugs. It gained popularity in late 1990s,the Laplace transforms has become the necessary element those who study mathematics, engineering, and for medical field. Its operations are multitudinous, without it numerous of our technological advances would have been suppressed, setting back the rapid-fire increase in technology ultramodern society has continued to bear substantiation to.

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