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Fractional Calculus Operators Involving the Product of Two Special Functions

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Abstract
In the present paper we establish fractional integration and differentiation formullas involving product of the finite classes of the classical orthogonal polynomials with the general class of multivariable polynomials.
Key words :Fractional Operators, Classical Orthogonal Polynomials,General Class Of Polynomials.

1. introduction

The fractional calculus operators involving various special functions has found considerable importance and applications in various sub-fields of applicable mathematical analysis. Many applications of fractional calculus can be found in turbulence and fluid dynamics, stochastic dynamical system, plasma physics and nonlinear control theory, image processing, nonlinear biological systems, astrophysics, and in quantum mechanics. Since last four decades, a number of researchers like Agarwal ([23], [22]), Agarwal and Jain [21], Baleanu [20], Baleanu and Mustafa [19], Baleanu et al. ([18],[17]), Kalla [16], Kalla and Saxena [15], Kilbas and Sebastian [14], Kiryakova ([13], [12]), Love [11], McBride [10], Purohit and Kalla [8] and Saigo [9]), so forth have studied, in depth, the properties, applications, and different extensions of various fractional calculus operators.shekhawat et al [7],[6] , Pandey et al [5] have studied the fractional derivative and fractional integral of product of special functions, respectivel.Large number of fractional integral formulas involving product of a variety of special functions have been developed by many authors ([4],[3],[2],[1]).

The generalization of the hypergeometric fractional integrals and derivative,including the Saigo operators has been introduced by Marichev[29] and later extended and studied by Saigo and Maeda[24](p.393)in term of any complex order with Appell function $F_3(.)$ in the kernel, as follows

Let $\varepsilon,\varepsilon',\delta,\delta',\gamma \epsilon C then for Re(\gamma)\geq 0$

$$\left(I_{0,x}^{\varepsilon,\varepsilon',\delta,\delta',\gamma} f\right)(x)=\frac{x^{-\varepsilon}}{\Gamma(\gamma)}\int_0^x (x-t)^{\gamma-1}t^{-\varepsilon'}F_3\left(\varepsilon,\varepsilon',\delta,\delta';\gamma;1-\frac{t}{x},1-\frac{x}{t}\right)f(t)dt \tag{1.1}$$

$$\left(I_{0,x}^{\varepsilon,\varepsilon',\delta,\delta',\gamma} f\right)(x)=\frac{x^{-\varepsilon'}}{\Gamma(\gamma)}\int_0^x (x-t)^{\gamma-1}t^{-\varepsilon}F_3\left(\varepsilon,\varepsilon',\delta,\delta';\gamma;1-\frac{t}{x},1-\frac{x}{t}\right)f(t)dt \tag{1.2}$$

Similarly for $Re(\gamma)\geq 0, k=[Re(\gamma)]+1$

$$\left(D_{0,x}^{\varepsilon,\varepsilon',\delta,\delta',\gamma} f\right)(x)=\left(I_{0,x}^{-\varepsilon',-\varepsilon,-\delta',-\delta,-\gamma} f\right)(x)$$

$$= \left(\frac{d}{dx}\right)^k \left(I_{0,x}^{-\varepsilon',-\varepsilon,-\delta'+k,-\delta,-\gamma+k} f\right)(x)$$
$$= \frac{1}{\Gamma(k-\gamma)}\left(\frac{d}{dx}\right)^k (x)^{\varepsilon'} \int_0^x (x-t)^{k-\gamma-1} t^{\varepsilon}$$
$$\times F_3\left(-\varepsilon',-\varepsilon,k-\delta',-\delta;k-\gamma;1-\tfrac{t}{x},1-\tfrac{x}{t}\right) f(t)dt, \quad (1.3)$$

And

$$\left(D_{x,\infty}^{\varepsilon,\varepsilon',\delta,\delta',\gamma} f\right)(x) = \left(I_{x,\infty}^{-\varepsilon',-\varepsilon,-\delta',-\delta,-\gamma} f\right)(x)$$
$$= \left(-\frac{d}{dx}\right)^k \left(I_{x,\infty}^{-\varepsilon',-\varepsilon,-\delta',-\delta+k,-\gamma+k} f\right)(x)$$
$$= \frac{1}{\Gamma(k-\gamma)}\left(\frac{d}{dx}\right)^k (x)^{\varepsilon} \int_x^\infty (t-x)^{k-\gamma-1} t^{\varepsilon'}$$
$$\times F_3\left(-\varepsilon',-\varepsilon,\delta',k-\delta;k-\gamma;1-\frac{x}{t},1-\frac{t}{x}\right) f(t)dt \quad (1.4)$$

where $F_3(.)$ denotes Appel function denoted as

$$F_3\left(\varepsilon,\varepsilon',\delta,\delta',\gamma;1-\frac{x}{t},1-\frac{t}{x}\right) = \sum_{m,n=0}^{\infty} \frac{(\varepsilon)_m(\varepsilon')_n(\delta)_m(\delta')_n}{(\gamma)_{m+n}} \frac{x^m}{m!}\frac{x^n}{n!}$$
$$(\max\{|x|,|y|\} < 1). \quad (1.5)$$

Power function formulas of fractional integral and derivative operators (1.1)-(1.4) are required for our present study as given in the following forms

$$\left(I_{0,x}^{\varepsilon,\varepsilon',\delta,\delta',\gamma} t^{\lambda-1}\right)(x) = \Gamma\begin{bmatrix}\lambda,\lambda+\gamma-\varepsilon-\varepsilon'-\delta,\lambda+\delta'-\varepsilon' \\ \lambda+\delta',\lambda+\gamma-\varepsilon-\varepsilon',\lambda+\gamma-\varepsilon'-\delta\end{bmatrix} x^{\lambda-\varepsilon-\varepsilon'+\gamma-1} \quad (1.6)$$

$$\mathrm{Re}(\gamma) > 0, (\mathrm{Re}(\lambda) > max\{0,\mathrm{Re}(\varepsilon+\varepsilon'+\delta-\gamma),\mathrm{Re}(\varepsilon'-\delta')\}),$$
$$\left(I_{x,\infty}^{\varepsilon,\varepsilon',\delta,\delta',\gamma} t^{\lambda-1}\right)(x) = \Gamma\begin{bmatrix}1-\lambda-\gamma+\varepsilon+\varepsilon',1-\lambda+\varepsilon+\delta'-\gamma,1-\lambda-\delta \\ 1-\lambda,1-\lambda+\varepsilon+\varepsilon'+\delta'-\gamma,1-\lambda+\varepsilon-\delta\end{bmatrix}$$
$$x^{\lambda-\varepsilon-\varepsilon'+\gamma-1} \quad (1.7)$$

$$(\mathrm{Re}(\gamma) > 0, \mathrm{Re}(\lambda) < min\{\mathrm{Re}(-\delta),\mathrm{Re}(\varepsilon+\varepsilon'-\gamma),\mathrm{Re}(\varepsilon+\delta'-\gamma)\})$$
$$\left(D_{0,x}^{\varepsilon,\varepsilon',\delta,\delta',\gamma} t^{\lambda-1}\right)(x) = \Gamma\begin{bmatrix}\lambda,\lambda-\gamma+\varepsilon+\varepsilon'+\delta',\lambda-\delta+\varepsilon \\ \lambda-\delta,\lambda-\gamma+\varepsilon+\varepsilon',\lambda-\gamma+\varepsilon+\delta'\end{bmatrix} x^{\lambda-\gamma+\varepsilon+\varepsilon'-1}, \quad (1.8)$$

$$(\mathrm{Re}(\lambda) > max\{0,\mathrm{Re}(\gamma-\varepsilon-\varepsilon'-\delta'),\mathrm{Re}(\delta-\varepsilon)\})$$
$$\left(D_{x,\infty}^{\varepsilon,\varepsilon',\delta,\delta',\gamma} t^{\lambda-1}\right)(x) = \Gamma\begin{bmatrix}1-\lambda+\delta',1-\lambda+\gamma-\varepsilon-\varepsilon',1-\lambda+\gamma-\varepsilon'-\delta \\ 1-\lambda,1-\lambda+\gamma-\varepsilon-\varepsilon'-\delta,1-\lambda-\varepsilon'+\delta'\end{bmatrix}$$
$$\times x^{\lambda-\gamma+\varepsilon+\varepsilon'-1} \quad (1.9)$$
$$(\Re(\lambda) < 1 + min\{\Re(\delta'),\Re(\gamma-\varepsilon-\varepsilon'),\Re(\gamma-\varepsilon'-\delta)\})$$

The symbol occurring in (1.6)-(1.9) is given by

$$\Gamma\begin{bmatrix}a,b,c \\ d,e,f\end{bmatrix} = \frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(d)\Gamma(e)\Gamma(f)} \quad (1.10)$$

An interesting further generalization of the generalized hyper-geometric function in a series representation is given by[25]

$$ {}_m\psi_n\left[\begin{matrix}(a_i,\alpha_i)_{1,m}\\ (b_j,\beta_j)_{1,n}\end{matrix}|z\right]=\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{m}\Gamma(a_i+k\alpha_i)}{\prod_{j=1}^{n}\Gamma(b_j+k\beta_j)}\frac{z^k}{k!} \tag{1.11}$$

Where $a_i, b_j \in C$ and $\alpha_i, \delta_j \in R, \alpha_i, \delta_j \neq 0; i = 1, \dots, m; j = 1, \dots, n$ and the asymptoticexpansionof ${}_m\psi_n$ for allvaluesoftheargument x, underthecondition

$$1+\sum_{j=1}^{q}\beta_j-\sum_{i=1}^{p}\alpha_i>0 \tag{1.12}$$

Also, were call the following multivariable eneralization of the polynomials $S_L^k(x)$which was considered by Srivastava and Garg [30]

$$S_L^{k_1,\dots,k_r}(x_1,\dots,x_r)=\sum_{k_1,\dots,k_r=0}^{h_1k_1+\cdots+h_rk_r\leq L}(-L)_{h_1k_1+\cdots+h_rk_r}A(L;k_1,\dots,k_r)\frac{x_1^{k_1}}{k_1!}\cdots\frac{x_r^{k_r}}{k_r!} \tag{1.13}$$

whereas the coefficient $A(L; k_1, \dots, k_r), (L, k_i \in N_0\{0\}, I = 1, \dots, r)$ are arbitrary chosen constants, real or complex. Clearly by setting $r = 1$ of the polynomialsdefined by (1.13) would correspond to the polynomials defined by Srivastava [29]. The solution to the differential equation

$$x(x+1)y_n''(x)+((2-p)x+(1+q))y_n'(x)-n(n-1+p)y_n(x)=0 \tag{1.14}$$

is given by the polynomial

$$M_n^{(p,q)}(x)=(-1)^n n!\sum_{k=0}^{\infty}\binom{p-n-1}{k}\binom{q+n}{n-k}(-x)^k \tag{1.15}$$

with respect to the weight function $W_{p,q}(x) = X_q(1 + x)^{-(p+q)}$The polynomials given in (1.15) are orthogonal on [0,1) if and only if $p > 2n + 1$ $andq > -1$. The polynomials $M_n^{(p,q)}$ can be related with hypergeometric functionsas

$$M_n^{(p,q)}(x)=(-1)^n n!\binom{q+n}{n}\;{}_2F_1(-n,n+1-p;q+1;-x) \tag{1.16}$$

Also the Jacobi polynomials $P_n^{(\varepsilon,\delta)}$ can be related as

$$M_n^{(p,q)}(x)=(-1)^n n!\,P_n^{(q,-p-q)}(2x+1)\Leftrightarrow P_n^{(p,q)}(x)=\frac{(-1)^n}{n!}M_n^{(-p-q,p)}\left(\frac{x-1}{2}\right) \tag{1.17}$$

Details related to this finite class of classical orthogonal polynomials can be found from ([26],[27],[28])

2 generalized fractional calculus operators involving product of srivastava polynomial and classic orthogonal polynomial

Theorm2.1t if $a_i, \varepsilon, \varepsilon', \delta, \delta', \gamma, \tau \in C$,$\mathrm{Re}(\gamma) > 0, Re(\tau) > \max\{0, Re(\varepsilon + \varepsilon' + \delta - \gamma), Re(\varepsilon - \delta')\}$,if the condition (1.12) is satisfied ,then the generalized fractional integration $I_{0,x}^{\varepsilon,\varepsilon',\delta,\delta',\gamma}$ of the product finite classes of the classical orthogonal polynomial and $M_n^{(p,q)}(.)$and the multivariable polynomials $S_L^{k_1,\dots,k_r}$is given by

$$\left(I_{0,x}^{\varepsilon,\varepsilon',\delta,\delta',\gamma} t^{\tau-1} S_L^{k_1,\dots,k_r}\left(a_1 t^{\lambda_1},\dots,a_r t^{\lambda_r}\right) M_n^{(p,q)}\left(t^{\xi}\right)\right)(x)$$

$$= (-1)^n \sum_{k_1,\dots,k_r=0}^{h_1k_1+\cdots+h_rk_r\le L} (-L)_{h_1k_1+\cdots+h_rk_r} A(L;k_1,\dots,k_r)\frac{a_1^{k_1}}{k_1!}\cdots\frac{a_r^{k_r}}{k_r!}$$
$$\times x^{\tau-\varepsilon-\varepsilon'+\gamma+\sum_{i=1}^r \lambda_i k_i-1}\frac{\Gamma(q+n+1)}{\Gamma(-n)\Gamma(1+n-p)}$$
$$\times {}_5\psi_4\left[\begin{matrix}(-n,1),(1+n-p,1),(\tau+\sum_{i=1}^r \lambda_i k_i,\xi)\\ (q+1,1),(\tau+\delta'+\sum_{i=1}^r \lambda_i k_i,\xi)\end{matrix}\right.$$
$$\left.\begin{matrix}\left(\tau+\gamma-\varepsilon-\varepsilon'-\delta+\sum_{i=1}^r \lambda_i k_i,\xi\right),\left(\tau+\delta'-\varepsilon'+\sum_{i=1}^r \lambda_i k_i,\xi\right)\\ \left(\tau+\gamma-\varepsilon-\varepsilon'+\sum_{i=1}^r \lambda_i k_i,\xi\right),\left(\tau+\gamma-\varepsilon'-\delta+\sum_{i=1}^r \lambda_i k_i,\xi\right)\end{matrix}\middle| -x^{\xi}\right] \tag{2.1}$$

LetΩ be the left side of (2.1) .Using (1.13) and (1.15), changing the order of the integration,and summation yields

$$\Omega = \sum_{k_1,\dots,k_r=0}^{h_1k_1+\cdots+h_rk_r\le L} (-L)_{h_1k_1+\cdots+h_rk_r} A(L;k_1,\dots,k_r)\frac{a_1^{k_1}}{k_1!}\cdots\frac{a_r^{k_r}}{k_r!}$$

$$\times (-1)^n n! \sum_{k=0}^{\infty}\binom{p-n-1}{k}\binom{q+n}{n-k}\left(I_{0,x}^{\varepsilon,\varepsilon',\delta,\delta',\gamma} t^{\tau+\xi k+\sum_{i=1}^r \lambda_i k_i-1}\right)(x)$$

For any $k=0,1,\dots,n$ since $Re(\tau+k)>Re(\tau)>\max\{0,Re(\varepsilon+\varepsilon'+\delta-\gamma),Re(\varepsilon'-\delta'\}$ and by applying (5.1.6), we obtain

$$\Omega = \sum_{k_1,\dots,k_r=0}^{h_1k_1+\cdots+h_rk_r\le L} (-L)_{h_1k_1+\cdots+h_rk_r} A(L;k_1,\dots,k_r)\frac{a_1^{k_1}}{k_1!}\cdots\frac{a_r^{k_r}}{k_r!}$$
$$\times (-1)^n \frac{\Gamma(q+n+1)}{\Gamma(-n)\Gamma(1+n-p)} x^{\tau-\varepsilon-\varepsilon'+\gamma+\sum_{i=1}^r \lambda_i k_i-1}$$
$$\times \sum_{k=0}^{\infty}\frac{\Gamma(1+n-p+k)\Gamma(-n+k)\Gamma(\tau+\sum_{i=1}^r \lambda_i k_i+\xi k)}{\Gamma(q+k+1)\Gamma(\tau+\delta'+\sum_{i=1}^r \lambda_i k_i+\xi k)}$$
$$\times \frac{\Gamma(\tau-\gamma-\varepsilon-\varepsilon'-\delta+\sum_{i=1}^r \lambda_i k_i+\xi k)\Gamma(\tau+\delta'-\varepsilon'+\sum_{i=1}^r \lambda_i k_i+\xi k)}{\Gamma(\tau+\gamma-\varepsilon-\varepsilon'+\sum_{i=1}^r \lambda_i k_i+\xi k)\Gamma(\tau+\gamma-\varepsilon'-\delta+\sum_{i=1}^r \lambda_i k_i+\xi k)}\frac{\left(-x^{\xi}\right)^k}{k!}, \tag{2.2}$$

interpreting the right-hand side of the above equation (2.2), in view of the definition (1.11), we arrive at the result (2.1).

Theorm2.2 $if\, a_i, \varepsilon, \varepsilon', \delta, \delta', \gamma, \tau \in C$, $\mathrm{Re}(\gamma) > 0, Re(1-\gamma -) < 1 + \min\{Re(-\delta), Re(\varepsilon + \varepsilon' - \gamma), Re(\varepsilon + \delta' - \gamma)$.if the condition (1.12) is satisfied ,then the gensralized fractional integration $I_{0,x}^{\varepsilon,\varepsilon',\delta,\delta',\gamma}$ of the product finite classes of the classical orthogonal polynomials $M_n^{(p,q)}(.)$ and multivariable polynomials $S_L^{k_1,\dots,k_r}(.)$is given by

$$\left(I_{x,\infty}^{\varepsilon,\varepsilon',\delta,\delta',\gamma} t^{-\gamma-\tau} S_L^{k_1,\dots,k_r}\left(a_1 t^{\lambda_1}, \dots, a_r t^{\lambda_r}\right) M_n^{(p,q)}\left(1/t^{\xi}\right)\right)(x)$$

$$= (-1)^n \sum_{k_1,\dots,k_r=0}^{h_1+\cdots+h_r k_r \le L} (-L)_{h_1 k_1+\cdots+h_r k_r} A(L; k_1, \dots, k_r) \frac{a_1^{k_1}}{k_1!} \cdots \frac{a_r^{k_r}}{k_r!}$$
$$\times\, x^{-\tau-\varepsilon-\varepsilon'+\sum_{i=1}^r \lambda_i k_i}$$
$$\times \frac{\Gamma(q+n+1)}{\Gamma(-n)\Gamma(1+n-p)} \quad {}_5\psi_4 \left[\begin{matrix} (-n,1),\ (1+n-p,1), (\varepsilon+\varepsilon'+\tau-\sum_{i=1}^r \lambda_i k_i, \xi) \\ (q+1,1), (\gamma+\tau-\sum_{i=1}^r \lambda_i k_i, \xi) \end{matrix}\right.$$
$$\left.\begin{matrix} (\varepsilon+\delta'+\tau-\sum_{i=1}^r \lambda_i k_i, \xi), (\gamma-\delta+\tau-\sum_{i=1}^r \lambda_i k_i, \xi) \\ (\varepsilon'+\delta'+\tau-\sum_{i=1}^r \lambda_i k_i, \xi), (\gamma+\varepsilon-\delta+\tau-\sum_{i=1}^r \lambda_i k_i, \xi) \end{matrix} \right| -\frac{1}{x^{\xi}} \right]$$

(2.3)

Proof On using (1.13) and (1.15), the left-hand side of (2.3) and changing the order of the integration and summation can be written as:

$$\Omega = \sum_{k_1,\dots,k_r=0}^{h_1 k_1+\cdots+h_r k_r \le L} (-L)_{h_1 k_1+\cdots+h_r k_r} A(L; k_1, \dots, k_r) \frac{a_1^{k_1}}{k_1!} \cdots \frac{a_r^{k_r}}{k_r!}$$

$$\times (-1)^n n! \sum_{k=0}^{\infty} \binom{p-n-1}{k} \binom{q+n}{n-k} \left(I_{x,\infty}^{\varepsilon,\varepsilon',\delta,\delta',\gamma} t^{-\gamma-\tau-\xi k+\sum_{i=1}^r \lambda_i k_i}\right)(x)$$

which on using the image formula (1.7), arrive at

$$\Omega = \sum_{k_1,\dots,k_r=0}^{h_1 k_1+\cdots+h_r k_r \le L} (-L)_{h_1 k_1+\cdots+h_r k_r} A(L; k_1, \dots, k_r) \frac{a_1^{k_1}}{k_1!} \cdots \frac{a_r^{k_r}}{k_r!} x^{-\tau-\varepsilon-\varepsilon'}$$
$$\times (-1)^n \frac{\Gamma(q+n+1)}{\Gamma(-n)\Gamma(1+n-p)} x^{\sum_{i=1}^r \lambda_i k_i}$$
$$\times \sum_{k=0}^{\infty} \frac{\Gamma(1+n-p+k)\Gamma(-n+k)\Gamma(\varepsilon+\varepsilon'+\tau-\sum_{i=1}^r \lambda_i k_i+\xi k)}{\Gamma(q+k+1)\Gamma(\gamma+\tau-\sum_{i=1}^r \lambda_i k_i+\xi k)}$$
$$\times \frac{\Gamma(\varepsilon+\delta'+\tau-\sum_{i=1}^r \lambda_i k_i+\xi k)\Gamma(\gamma-\delta+\tau-\sum_{i=1}^r \lambda_i k_i+\xi k)}{\Gamma(\varepsilon+\varepsilon'+\delta'+\tau-\sum_{i=1}^r \lambda_i k_i+\xi k)\Gamma(\gamma+\varepsilon-\delta+\tau-\sum_{i=1}^r \lambda_i k_i+\xi k)} \frac{\left(-x^{-\xi}\right)^k}{k!}$$

(2.4)

Interpreting the right-hand side of(2.4),in view o f the efinition (1.11), we arrive at the result (2.3).

Corollary2.1. Let, $\mathrm{Re}(\gamma) > 0, Re(1-\gamma -) < 1 + \min\{Re(-\delta), Re(\varepsilon+\varepsilon'-\gamma), Re(\varepsilon+\delta'-\gamma)\}$, If we reduce $S_L^{k_1,\dots,k_r}$ to unity 1 in (2.1),then the generalized fractional integration $I_{0,x}^{\varepsilon,\varepsilon',\delta,\delta',\gamma}$ of the finite classes of the classical orthogonal polynomials $M_n^{(p,q)}$ is given by

$$\left(I_{0,x}^{\varepsilon,\varepsilon',\delta,\delta',\gamma} t^{\tau-1} M_n^{(p,q)}\left(t^{\xi}\right)\right)(x) = (-1)^n x^{\tau-\varepsilon-\varepsilon'+\gamma-1} \frac{\Gamma(q+n+1)}{\Gamma(-n)\Gamma(1+n-p)}$$
$$\times {}_5\psi_4\left[\begin{matrix}(-n,1),(1+n-p,1),(\tau,\xi),(\tau+\gamma-\varepsilon-\varepsilon'-\delta,\xi),(\tau+\delta'-\varepsilon',\xi)\\(q+1,1),(\tau+\delta',\xi),(\tau+\gamma-\varepsilon-\varepsilon',\xi),(\tau+\gamma-\varepsilon'-\delta,\xi)\end{matrix}\middle| -x^{\xi}\right]$$
(2.5)

Corollar.2.2 let $\mathrm{Re}(\gamma) > 0, Re(\tau) > \max\{0, Re(\varepsilon+\varepsilon'+\delta-\gamma), Re(\varepsilon'-\delta')\}$)
If we reduce $S_L^{k_1,\dots,k_r}$ to unity 1 in (2.3) then the generalized fractional integration $I_{x,\infty}^{\varepsilon,\varepsilon',\delta,\delta',\gamma}$ of the finite classes of the classical orthogonal polynomials $M_n^{(p,q)}$ is given by

$$\left(I_{x,\infty}^{\varepsilon,\varepsilon',\delta,\delta',\gamma} t^{-\gamma-\tau} M_n^{(p,q)}\left(1/t^{\xi}\right)\right)(x) = (-1)^n \frac{\Gamma(q+n+1)}{\Gamma(-n)\Gamma(1+n-p)} x^{-\tau-\varepsilon-\varepsilon'}$$
$$\times {}_5\psi_4\left[\begin{matrix}(-n,1),(1+n-p,1),(\tau,\xi),(\tau+\gamma-\varepsilon-\varepsilon'-\delta,\xi),(\tau+\delta'-\varepsilon',\xi)\\(q+1,1),(\tau+\delta',\xi),(\tau+\gamma-\varepsilon-\varepsilon',\xi),(\tau+\gamma-\varepsilon'-\delta,\xi)\end{matrix}\middle| -x^{\xi}\right]$$
(2.6)

Theorem2.3. let $\varepsilon, \varepsilon', \delta, \delta', \gamma, \tau, \in C$ such that $Re(\gamma) > 0$, Further,let the constants satisfy the condition $a_i, b_j \in C, and \varepsilon_i, \delta_j \in R,$ and $\varepsilon_i, \delta_j \neq 0; i = 1, \dots, p; j = 1, \dots, q$ such that condition (1.12) is also satisfied.
Then the generalized fractional derivative of the product finite class of the classical orthogonal polynomials and multivariable polynomials is given by

$$\left(D_{0,x}^{\varepsilon,\varepsilon',\delta,\delta',\gamma} t^{\tau-1} S_L^{k_1,\dots,k_r}\left(a_1 t^{\lambda_1}, \dots, a_r t^{\lambda_r}\right) M_n^{(p,q)}\left(t^{\xi}\right)\right)(x)$$
$$= (-1)^n \sum_{k_1 k_1+\cdots+h_r k_r \leq L}^{\sum_{1,\dots,k_r=0}^{k}} (-L)_{h_1 k_1+\cdots+h_r k_r} A(L; k_1, \dots, k_r) \frac{a_1^{k_1}}{k_1!} \cdots \frac{a_r^{k_r}}{k_r!}$$
$$\times x^{\tau+\varepsilon+\varepsilon'-\gamma+\sum_{i=1}^r \lambda_i k_i - 1} \frac{\Gamma(q+n+1)}{\Gamma(-n)\Gamma(1+n-p)}$$

$$\times {}_5\psi_4\left[\begin{matrix}(-n,1),(1+n-p,1),(\tau+\sum_{i=1}^r \lambda_i k_i, \xi)\\(q+1,1),(\tau-\delta+\sum_{i=1}^r \lambda_i k_i, \xi)\end{matrix}\right.$$
$$\left.\begin{matrix}(\tau-\gamma+\varepsilon+\varepsilon'+\delta'+\sum_{i=1}^r \lambda_i k_i, \xi),(\tau-\delta+\varepsilon+\sum_{i=1}^r \lambda_i k_i, \xi)\\(\tau-\gamma+\varepsilon+\varepsilon'+\sum_{i=1}^r \lambda_i k_i, \xi),(\tau-\gamma+\varepsilon+\delta'+\sum_{i=1}^r \lambda_i k_i, \xi)\end{matrix}\middle| -x^{\xi}\right]$$
(2.7)

Proof On using (1.13) and (1.15), writing the function in the series form, the left-hand side of (2.7), le

$$\Omega = \sum_{k_1,\dots,k_r=0}^{h_1 k_1+\cdots+h_r k_r \leq L} (-L)_{h_1 k_1+\cdots+h_r k_r} A(L; k_1, \dots, k_r) \frac{a_1^{k_1}}{k_1!} \cdots \frac{a_r^{k_r}}{k_r!}$$
$$\times (-1)^n n! \sum_{k=0}^{\infty} \binom{p-n-1}{k} \binom{q+n}{n-k} \left(I_{0,x}^{-\varepsilon',-\varepsilon,-\delta',-\delta,-\gamma} t^{\tau+\xi k+\sum_{i=1}^r \lambda_i k_i - 1}\right)(x)$$

Now upon using the image formulas (1.8), which is valid under the condition stated with Theorem 2.3 we obtain

$$\Omega = (-1)^n \sum_{k_1,\dots,k_r=0}^{h_1k_1+\cdots+h_rk_r \le L} (-L)_{h_1k_1+\cdots+h_rk_r} A(L;k_1,\dots,k_r)\frac{a_1^{k_1}}{k_1!}\cdots\frac{a_r^{k_r}}{k_r!}$$
$$\times x^{\tau-\gamma+\varepsilon+\varepsilon'+\sum_{i=1}^r \lambda_i k_i - 1}\frac{\Gamma(q+n+1)}{\Gamma(-n)\Gamma(1+n-p)}$$
$$\times \sum_{k=0}^{\infty} \frac{\Gamma(1+n-p+k)\Gamma(-n+k)\Gamma(\tau+\sum_{i=1}^r \lambda_i k_i+\xi k)}{\Gamma(q+k+1)\Gamma(\tau-\delta+\sum_{i=1}^r \lambda_i k_i+\xi k)}$$
$$\times \frac{\Gamma(\tau-\gamma+\varepsilon+\varepsilon'+\delta'+\sum_{i=1}^r \lambda_i k_i+\xi k)\Gamma(\tau-\delta+\varepsilon+\sum_{i=1}^r \lambda_i k_i+\xi k)}{\Gamma(\tau-\gamma+\varepsilon+\varepsilon'+\sum_{i=1}^r \lambda_i k_i+\xi k)\Gamma(\tau-\gamma+\varepsilon+\delta'+\sum_{i=1}^r \lambda_i k_i+\xi k)}\frac{(-x^{\xi})^k}{k!} \tag{2.8}$$

Interpreting the right hand side of the above equation(2.8) ,In view of the definition (1.11), we arrive at the result(2.7).

Theorem 2.4.$if\, a_i, \varepsilon, \varepsilon', \delta, \delta', \gamma, \tau \in C,$ The generalized fractional derivative $D_{x,\infty}^{\varepsilon,\varepsilon',\delta,\delta',\gamma}$ of the product finite classes of the classical orthogonal polynomials $M_n^{(p,q)}$and multi variable polynomials givenby $S_L^{k_1,\dots,k_r}$

$$\left(D_{x,\infty}^{\varepsilon,\varepsilon',\delta,\delta',\gamma} t^{\gamma-\tau} S_L^{k_1,\dots,k_r}(a_1t^{\lambda_1},\dots,a_rt^{\lambda_r}) M_n^{(p,q)}(1/t^{\xi})\right)(x)$$
$$= (-1)^n \sum_{k_1k_1+\cdots,k_r=0}^{k_rk_r \le L} (-L)_{h_1k_1+\cdots+h_rk_r} A(L;k_1,\dots,k_r)\frac{a_1^{k_1}}{k_1!}\cdots\frac{a_r^{k_r}}{k_r!}$$
$$\times x^{-\tau+\varepsilon+\varepsilon'+\sum_{i=1}^r \lambda_i k_i}\frac{\Gamma(q+n+1)}{\Gamma(-n)\Gamma(1+n-p)}$$
$$\times_5 \psi_4\left[\begin{matrix}(-n,1),(1+n-p,1),(\tau-\varepsilon-\varepsilon'-\sum_{i=1}^r \lambda_i k_i,\xi)\\ (q+1,1),(\tau-\gamma-\sum_{i=1}^r \lambda_i k_i,\xi)\end{matrix}\right.$$
$$\left.\begin{matrix}(\tau-\varepsilon'-\delta-\sum_{i=1}^r \lambda_i k_i,\xi),(\tau-\gamma+\delta'-\sum_{i=1}^r \lambda_i k_i,\xi)\\ (\tau-\varepsilon-\varepsilon'-\delta-\sum_{i=1}^r \lambda_i k_i,\xi),(\tau-\gamma-\varepsilon'+\delta'-\sum_{i=1}^r \lambda_i k_i,\xi)\end{matrix}\right| -\frac{1}{x^{\xi}}\right] \tag{2.9}$$

Proof*:* On using(1.13)and(1.15),theleft-handsideof(2.9)canbewrittenas:

$$\Omega = \sum_{k_1,\dots,k_r=0}^{h_1k_1+\cdots+h_rk_r \le L} (-L)_{h_1k_1+\cdots+h_rk_r} A(L;k_1,\dots,k_r)\frac{a_1^{k_1}}{k_1!}\cdots\frac{a_r^{k_r}}{k_r!}$$
$$(-1)^n n! \sum_{k=0}^{\infty} \binom{p-n-1}{k}\binom{q+n}{n-k}\left(I_{x,\infty}^{-\varepsilon',-\varepsilon,-\delta',-\delta,-\gamma} t^{\gamma-\tau-\xi k+\sum_{i=1}^r \lambda_i k_i}\right)(x)$$

which on using the image formula (1.9), we arrive at

$$\Omega = \sum_{k_1,\dots,k_r=0}^{\hbar_1k_1+\cdots+\hbar_rk_r \le L} (-L)_{\hbar_1k_1+\cdots+\hbar_rk_r} A(L;k_1,\dots,k_r)\frac{a_1^{k_1}}{k_1!}\cdots\frac{a_r^{k_r}}{k_r!} x^{\tau+\varepsilon+\varepsilon'}$$
$$\times (-1)^n \frac{\Gamma(q+n+1)}{\Gamma(-n)\Gamma(1+n-p)} x^{\sum_{i=1}^r \lambda_i k_i}$$
$$\times \sum_{k=0}^{\infty} \frac{\Gamma(1+n-p+k)\Gamma(-n+k)\Gamma(\tau-\varepsilon-\varepsilon'-\sum_{i=1}^r \lambda_i k_i+\xi k)}{\Gamma(q+k+1)\Gamma(\tau-\gamma-\sum_{i=1}^r \lambda_i k_i+\xi k)}$$

$$\frac{\Gamma(\tau-\varepsilon^{'}-\delta-\sum_{i=1}^{r}\lambda_i k_i+\xi k)\Gamma(\tau-\gamma+\delta^{'}-\sum_{i=1}^{r}\lambda_i k_i+\xi k)}{\Gamma(\tau-\varepsilon-\varepsilon^{'}-\delta-\sum_{i=1}^{r}\lambda_i k_i+\xi k)\Gamma(\tau-\gamma-\varepsilon^{'}+\delta^{'}-\sum_{i=1}^{r}\lambda_i k_i+\xi k)}$$
$$\frac{\left(-x^{-\xi}\right)^k}{k!}$$
(2.10)

Interpreting the right-hand side of (2.10), in view of the definition (1.11), we arrive at the result (2.9).

Corollary2.3. If we reduce $S_L^{k_1,\dots,k_r}$ to unity 1 in (2.7) .the The generalized fractional derivative $D_{0,x}^{\varepsilon,\varepsilon^{'},\delta,\delta^{'},\gamma}$ of the product of finite classes of the classical orthogonal polynomials $M_n^{(p,q)}$ is given by

$$\left(D_{0,x}^{\varepsilon,\varepsilon^{'},\delta,\delta^{'},\gamma}t^{\tau-1}M_n^{(p,q)}\left(t^{\xi}\right)\right)(x)=(-1)^n\frac{\Gamma(q+n+1)}{\Gamma(-n)\Gamma(1+n-p)}x^{\tau+\varepsilon+\varepsilon^{'}-\gamma-1}$$
$$\times\quad {}_5\psi_4\left[\begin{matrix}(-n,1),(1+n-p,1),(\tau,\xi),(\tau-\gamma+\varepsilon+\varepsilon^{'}+\delta^{'},\xi),(\tau-\delta+\varepsilon,\xi)\\(q+1,1),(\tau-\delta,\xi),(\tau-\gamma+\varepsilon+\varepsilon^{'},\xi),(\tau-\gamma+\varepsilon+\delta^{'},\xi)\end{matrix}\middle|-x^{\xi}\right]$$
(2.11)

Corollary2.4. . If we reduce $S_L^{k_1,\dots,k_r}$ to unity 1 in (2.9) .then The generalized fractional derivative $D_{x,\infty}^{\varepsilon,\varepsilon^{'},\delta,\delta^{'},\gamma}$ of the product of finite classes of the classical orthogonal polynomials $M_n^{(p,q)}$ is givenby

$$\left(D_{x,\infty}^{\varepsilon,\varepsilon^{'},\delta,\delta^{'},\gamma}t^{\gamma-\tau}M_n^{(p,q)}\left(1/t^{\xi}\right)\right)(x)=(-1)^n\frac{\Gamma(q+n+1)}{\Gamma(-n)\Gamma(1+n-p)}x^{-\tau+\varepsilon+\varepsilon^{'}}$$

$$\times_5\,\psi_4\left[\begin{matrix}(-n,1),(1+n-p,1),(\tau-\varepsilon-\varepsilon^{'},\xi),(\tau-\varepsilon^{'}-\delta,\xi),(\tau-\gamma+\delta^{'},\xi)\\(q+1,1),(\tau-\gamma,\xi),(\tau-\varepsilon-\varepsilon^{'}-\delta,\xi),(\tau-\gamma-\varepsilon^{'}+\delta^{'},\xi)\end{matrix}\middle|-\frac{1}{x^{\xi}}\right]$$
(2.12)

3 Special cases

COROLLARY3.11if we If we put $\varepsilon=\varepsilon+\delta,\varepsilon^{'}=0,\delta=-\eta$and $and\gamma=\varepsilon$in theorm 2.1then for x>0 the following formula holds true

$$\left(I_{0,x}^{\varepsilon,\delta,\eta}t^{\tau-1}S_L^{k_1,\dots,k_r}\left(a_1t^{\lambda_1},\dots,a_rt^{\lambda_r}\right)M_n^{(p,q)}\left(t^{\xi}\right)\right)(x)$$

$$=(-1)^n\sum_{k_1,\dots,k_r=0}^{\hbar_1+\cdots+\hbar_r k_r\le L}(-L)_{\hbar_1k_1+\cdots+\hbar_rk_r}A(L;k_1,\dots,k_r)\frac{a_1^{k_1}}{k_1!}\cdots\frac{a_r^{k_r}}{k_r!}$$

$$\times x^{\tau-\delta+\sum_{i=1}^{r}\lambda_ik_i-1}\frac{1(q+n+1)}{\Gamma(-n)\Gamma(1+n-p)}$$

$${}_4\psi_3\left[\begin{matrix}(-\mathrm{n},1),(1+\mathrm{n}-\mathrm{p},1)(\tau-\delta+\eta+\sum_{i=1}^{r}\lambda_ik_i,\xi),(\tau+\sum_{i=1}^{r}\lambda_ik_i,\xi)\\(q+1,1),(\tau-\delta+\sum_{i=1}^{r}\lambda_ik_i,\xi),(\tau+\varepsilon+\eta+\sum_{i=1}^{r}\lambda_ik_i,\xi)\end{matrix}\middle|-x^{\xi}\right]$$
(3.1)

COROLLARY3.2 If we put $\varepsilon=\varepsilon+\delta,\varepsilon^{'}=0,\delta=-\eta$and $and\gamma=\varepsilon$ in theorm 2.2.then for $x>0$, the following formula holds true

$$\left(I_{x,\infty}^{\varepsilon,\delta,\eta}t^{-\gamma-\tau}S_L^{k_1,\dots,k_r}\left(a_1t^{\lambda_1},\dots,a_rt^{\lambda_r}\right)M_n^{(p,q)}\left(1/t^{\xi}\right)\right)(x)$$

$$= (-1)^n \sum_{k_1,\dots,k_r=0}^{\hbar_1 k_1+\cdots+\hbar_r k_r \le L} (-L)_{\hbar_1 k_1+\cdots+\hbar_r k_r} A(L;k_1,\dots,k_r)\frac{a_1^{k_1}}{k_1!}\cdots\frac{a_r^{k_r}}{k_r!}$$

$$\times\, x^{\tau-\delta+\sum_{i=1}^{r}\lambda_i k_i}\,\frac{\Gamma(q+n+1)}{\Gamma(-n)\Gamma(1+n-p)}{}_4\psi_3\left[\begin{matrix}(-n,1),(1+n-p,1)\\(q+1,1),\end{matrix}\right.$$

$$\left.\begin{matrix}\left(\varepsilon+\delta+\tau-\sum_{i=1}^{r}\lambda_i k_i,\xi\right),\left(\varepsilon+\eta+\tau-\sum_{i=1}^{r}\lambda_i k_i,\xi\right)\\\left(\varepsilon+\tau-\sum_{i=1}^{r}\lambda_i k_i,\xi\right),\left(2\varepsilon+\delta+\eta+\tau-\sum_{i=1}^{r}\lambda_i k_i,\xi\right)\end{matrix}\middle|-x^{\xi}\right]$$

(3.2

4 Conclusion

Thus ,the generalized fractional integral and derivative operators derived in this paper are capable of being applied to diverse polynomial systems in one ,two ,and more variables.By assigning appropriate special values to the coefficient occurring in the definition(1.13) ,the polynomials can be reduced not only to the different classical orthogonal polynomials such as the Jacobi polynomial, Hermite polynomials and Leguerre polynomials but also to the Bessel polynomials, the generalized heat polynomials, the Konhauser biorthogonal polynomials, the generalized hypergeometric polynomials of Bateman, Brafmen, Fesenmyer, Gould-Hopper, Rice, Sylvester, and soon.

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