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Fixed Point Theorem Apply to Modified Lipschitzcondition to Solve Differential Equations

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ABSTRACT

The goal of this work is to look at generalised contractions and show that they may be used to derive fixed point theorems in generalised metric spaces. Furthermore, we'll also use fixed point theorems to show that the outcome of the ordinary differential equation exists and is exclusive (ODE). Fixed point theorem applies to modified Lipschitz condition. First, we have proved that generalised condition of Boyd Wong theorem and using this concept we have modified Lipschitz condition by adding constant which is suitable for solving differential equation.

Keywords: Fixed point theorem, Metric space, Boundary value problem, differential equations, Integral equation.

INTRODUCTION:

There is a theorem of several fixed points such as Banach fixed point theorem, Brouwer fixed point theorem, Borel fixed point theorem, Schauder fixed point theorem. Using the whole theorem of fixed points, we can solve different calculations. a fixed point theorem is suitable for establishing local availability and diversity of ODE solutions. Numerical research is a broad field of pure and applied statistics. Finding a solution to a theoretical or real problem compares to the presence of a fixed point of the relevant map or operator in a broader mathematical, economic, modelling, and engineering challenge. As a result, static points are important in many fields of mathematics, science, and engineering. The fixed point theorem is used to prove that an important equation has a solution. It is now a worldwide tool for solving problems in many areas of mathematical analysis because of its simplicity and practicality. We have proven the point of focus centered on the complete metric space showing the close resemblance to Boyd's theories, Wong. The Banach fixed point theorem is important as the basis for existence and distinct theory in several branches of analysis. In this way, the theorem serves as an inspiring example of the power of effective analytical methods combined with the use of consistent point methods.

Definition 1.1. Fixed point. It is a point in the field that mapped to itself.That is, Function F of a set X into itself such that, a point $c \in X$ such that $f(c) = c$.

<u>Definition</u>1.2 (Metric Space) [4]: A "Metric Space" is a pair (X,d), where X is a set and d is a metric on X (distance function on X), that is, a function defined on $X \times X$ such that for all $x, y, z \in X$ we have:

(1) d is real-valued, finite and non-negative,
(2)d(x,y) =0if and only if, x = y
(3)d(x,y) = d(x,y)
(4)d(x,y) ≤d(x,z) + d(z,y)

<u>Definition</u> 1.2: Let (X,d) be a metric space and let P be a mapping on X. Then P is called a "Contraction" if there exists $r \in [0,1)$ such that $d(Px, Py) \leq rd(x,y)$for all $x, y \in X$.

<u>Definition</u> 1.3 Let g be continuous function on a rectangle $R = \{(t,x)/|t-t_0| \leq a, |x-x_0| \leq b\}$ and thus bounded on R, say$|g(t,x)| \leq c$ for all (t , x) ∈ R.Suppose that g satisfies a Lipschitz condition on R with respect to its second argument, that is, there is a constant k (Lipschitz constant) such that for (t, x), (t, v) ∈R $|g(t,x) - g(t,v)| \leq k|x - v|$

<u>Definition</u> 1.4.A order$\{x_n\}_{n=1}^{\infty}$ in a metric space (X,d) is said to converge or to be convergent if there is an $x \in X$ such that $\lim_{n\to\infty} d(x_n, x) = 0$, x is called the limit of $\{x_n\}_{n=1}^{\infty}$ and we write limit as x tend to infinity and x_n =x orx_n tends to x as n tends to infinity.

<u>Definition</u>1.5:A sequence $\{x_n\}_{n=1}^{\infty}$ in a metric space (X,d) is said to be a "Cauchy Sequence" if for every $\varepsilon > 0$there is an $N = N(\varepsilon)$ such that $d(x_m, x_n) < \varepsilon$ for every $m, n \geq N$

<u>Definition</u> 1.6:A "Fixed Point" of a mapping $P: Y \to Y$is an y € Y which is mapped onto itself, that is $Py = y$.

<u>Theorem</u> 1.1.[9]Banach fixed point theorem. Let(X,d)be a complete metric space and let F be a contraction on X. Then F has a unique fixed point.

<u>Theorem</u> 1.2. Schauder's fixed point Theorem. Let X be a Banach space M is subset of X be nonempty, convex, bounded, closed and P: X → M be a compact operator. Then T has unique fixed point

<u>Theorem</u> 1.3.Brouwer's fixed point theorem. It states that for any continuous function f mapping a compact convex set to itself there is a point x such that g(x)=x

<u>Theorem</u>1.4 (Boyd–Wong Theorem) [3]: Let (X,d)be a complete metric space,and suppose that $T: X \to X$ satisfies$d(Px, Py) \leq \theta(d(x,y))$, for all$x, y \in X$, where$\theta: \mathbb{R} \to [0, \infty)$is upper semi-continuous from the exact (that is, for anysequencet_n , $t \geq 0 \Rightarrow \limsup_{n\to\infty} \theta(t_n) \leq \theta(t)$)and satisfies$0 < \theta(t) \leq t$ for $t > 0$. Then, T has a unique fixed point.

<u>Theorem</u>1.5 [16]: Let (X,d) be a complete metric space and suppose that $P: X \to X$ satisfies $d(Px, Py) \leq \theta(d(x,y))$ for all $x, y \in X$, where, $\theta: (0, \infty) \to (0, \infty)$ is monotone non-decreasing and satisfies $\lim_{n\to\infty} \theta^n(t) = 0$ for all $t > 0$. Then P has a unique fixed point in X.

Main Result1

<u>Theorem</u>2.1: Let (X,d) there be a complete metric space and say$P: X \to X$ satisfies $d(Px, Py) \leq \alpha\theta(d(x, Px)) + \beta\theta(d(y, Py)) + \gamma\theta(d(x, y))$ for all $x, y \in X$ (A)

Where, $\theta: \mathbb{R} \to [0, \infty)$is the highest point running from the exact and satisfies$0 \leq \theta(t) < t$ for all $t > 0$, $\theta(0) = 0$. Also $0 < \alpha + \beta + \gamma < 1, \alpha > 0, \beta > 0, \gamma > 0$. Then P has a fixed point that is unique to X

<u>Proof</u>: Let $x_0 \in X$ be a random but unchanging part of X. Define multiplication sequence with $\{x_n\}_{n=1}^{\infty}$ in X

$$x_1 = Px_0, x_2 = Px_1 = P^2x_0, x_3 = Px_2 = P^3x_0, \ldots\ldots, x_n = Px_{n-1} = P^nx_0, \ldots\ldots$$

By the condition (A) on P we get,

$$\begin{aligned}
d(x_n, x_{n+1}) &= d(Px_{n-1}, Px_n) \\
&\leq \alpha\theta(d(x_{n-1}, Px_{n-1})) + \beta\theta(d(x_n, Px_n)) + \gamma\theta(d(x_{n-1}, x_n)) \\
&= \alpha\theta(d(x_{n-1}, x_n)) + \beta\theta(d(x_n, x_{n+1})) + \gamma\theta(d(x_{n-1}, x_n)) \\
&< \alpha d(x_{n-1}, x_n) + \beta d(x_n, x_{n+1}) + \gamma d(x_{n-1}, x_n) (\because \psi(t) < t)
\end{aligned}$$

$$d(x_n, x_{n+1}) < \alpha d(x_{n-1}, x_n) + \beta d(x_n, x_{n+1}) + \gamma d(x_{n-1}, x_n)$$

$$\therefore (1 - \beta) d(x_{n+1}, x_n) < (\alpha + \gamma) d(x_{n-1}, x_n)$$

$$\therefore d(x_{n+1}, x_n) < \frac{\alpha + \gamma}{1 - \beta} d(x_{n-1}, x_n)$$

$$\therefore d(x_{n+1}, x_n) < k d(x_{n-1}, x_n)$$

Where, $k = \frac{\alpha+\gamma}{1-\beta}$. Here $0 < k < 1$ because $0 < \alpha + \beta + \gamma < 1, \alpha > 0, \beta > 0, \gamma > 0$.

Enduring in this way, we get $d(x_{n+1}, x_n) < k^n d(x_0, x_1)$. Taking limit as $n \to \infty$ we get,

$$d(x_{n+1}, x_n) \to 0 (\because 0 < k < 1)$$

Then$\{x_n\}_{n=1}^{\infty}$there is the Cauchy sequencein X. As X is a complete metric space, there exists $x \in X$ such that $\lim_{n \to \infty} x_n = x$. We will show that that is the fixed point of P.

As P a is continuous function we have,

$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} Px_{n-1} = P\left(\lim_{n\to\infty} x_{n-1}\right) = Px.$

Therefore $Px = x$and $x$is a fixed point of P.

Next,we will highlight that unique P point.

Let $y \in X$ be another fixed point of P.

Again,with the form (A) we receive,

$$\begin{aligned}
d(x, y) &= d(Px, Py) \leq \alpha\theta(d(x, Px)) + \beta\theta(d(y, Py)) + \gamma\theta(d(x, y)) \\
&= \alpha\theta(d(x, x)) + \beta\theta(d(y, y)) + \gamma\theta(d(x, y)) \\
&= \alpha\theta(0) + \beta\theta(0) + \gamma\theta(d(x, y)) \\
&= \gamma\theta(d(x, y)) \\
&< \gamma d(x, y)
\end{aligned}$$

Thus,$d(x, y) < \gamma d(x, y)$

This is only conceivable if and when $d(x, y) = 0$, because $\gamma < 1$. Thus $x = y$ As a result, P's fixed point is unique.

Example 2.1: Deliberate the whole metric space of all non-negative real numbers with modulus value metric. Suppose that$P: X \to X$defined by $Px = \frac{x}{8}$. Let, $\theta: R \to [0, \infty)$ is defined by $\theta(t) = \frac{t}{2}$. The function $\theta(t)$ is continuous (and hence upper semi-continuous from right), also $0 < \theta(t) < t$ for all $t > 0$, $\theta(0) = 0$. Let $\alpha = \beta = \gamma = \frac{1}{4}$.

Then clearly $0 < \alpha + \beta + \gamma = \frac{3}{4} < 1, \alpha > 0, \beta > 0, \gamma > 0.$ Weprove that the condition (A) of the theorem 2.1 is satisfied.

We observe that $d(Px, Py) = d\left(\frac{x}{8}, \frac{y}{8}\right) = \frac{|x-y|}{8}$.

Also

$$\begin{aligned}
&\alpha d(x, Px) + \beta d(y, Py) + \gamma d(x, y) \\
&= \frac{1}{4}\theta\left(d\left(x, \frac{x}{8}\right)\right) + \frac{1}{4}\theta\left(d\left(y, \frac{y}{8}\right)\right) + \frac{1}{4}\theta(d(x, y)) \\
&= \frac{1}{4}\theta\left(\frac{7x}{8}\right) + \frac{1}{4}\theta\left(\frac{7y}{8}\right) + \frac{1}{4}\theta(|x-y|) \\
&= \frac{1}{4}\frac{\left(\frac{7x}{8}\right)}{2} + \frac{1}{4}\frac{\left(\frac{7y}{8}\right)}{2} + \frac{1}{4}\frac{|x-y|}{2} = \frac{7x}{64} + \frac{7y}{64} + \frac{|x-y|}{8} \\
&= \frac{7(x+y)}{64} + \frac{|x-y|}{8}.
\end{aligned}$$

Thus obviously

$d(Px, Py) = d\left(\frac{x}{8}, \frac{y}{8}\right) = \frac{|x-y|}{8} < \frac{7(x+y)}{64} + \frac{|x-y|}{8} = \alpha d(x, Px) + \beta d(y, Py) + \gamma d(x, y)$ for all $x \in R^+$. The condition (A) of the theorem 2.1 is satisfied. We see that $x = 0$ is the unique fixed point of the function P.

Lemma2.1 [7]For ordinary differential equations the contraction mapping theorem can be used to prove the existence and uniqueness of theinitial problem. We consider a first-order of ODEs for afunction u(t) that take value inR^n

u is solution of $u'(t) = g(t, u(t))$ -------(2.1) satisfying the initial condition $u(t_0) = u_0$

The initial value problem can be reformulated as an integral equation

$u(t) = u_0 + \int_{t_0}^{t} g(s, u(s))ds$ -- (2)

By fundamental theorem of calculus, a continuous solution of (2) is continuously differentiable solution of (1). Equation (2) may be written as fixed point equation $u = Pu$

The map P is defined by

$$Pu(t) = u_0 + \int_{t_0}^{t} g(s, u(s))ds$$

Definition 2.1: Suppose that $g : I \times R^n \to R^n$ where I is interval in R.We say that g (t,u(t)) is globally continuous function of u uniformly in t if there is constant K > 0 such that $|g(t, x) - g(t, v)| \le k|x - v|$ for all x,y ∈ R (2.2)

Definition2.2. Suppose that P satisfies a modifiedLipschitz condition on R with respect to its second argument, that is, there is a constant α, β, γ (Lipschitz constant) such that for (t, x), (t, v) ∈R

$$|P(t,x) - P(t,v)| \leq \alpha\,|x - v| + \beta|x - Px| + \gamma|v - Pv|, \qquad (2.3)$$

Also $0 < \alpha + \beta + \gamma < 1, \alpha > 0, \beta > 0, \gamma > 0$.

Main Results 2

Theorem 2.1: Assume g: I × R $^n \to R^n$ where I is interval in R and t_0 is the point within I. If$g(t,u)$is continuous function of (t,u) and modifiedLipschitz condition with continuous function of u uniformly in t on I × R ^{n}, then there is unique continuously differentiable function u:I → R^n that satisfies (2.1)

Proof: We will show that P is a contraction on the space of continuous function defined on a

time interval$t_0 \leq t \leq t_0 + \delta$ for sufficiently small δ

Suppose that $u, v\colon [t_0, t_0 + \delta] \to R^n$ are two continuous functions then form

$$|Pu - Pv| = \sup_{t_0 \leq t \leq t_0 + \delta} |Pu(t) - Pv(t)|$$

$$\leq \sup_{t_0 \leq t \leq t_0 + \delta} |Pu(t) - u(t) + u(t) - v(t) + v(t) - Pv(t)|$$

$$\leq \sup_{t_0 \leq t \leq t_0 + \delta} \left| u_0 + \int_{t_0}^{t} g(s,u(s))ds - u_0 - \int_{t_0}^{t} g(s,u(s))ds + u_0 + \int_{t_0}^{t} g(s,u(s))ds \right.$$
$$- \qquad v_0 - \int_{t_0}^{t} g(s,v(s))ds + v_0 + \int_{t_0}^{t} g(s,v(s))ds - u_0$$
$$\left. - \int_{t_0}^{t} g(s,v(s))ds \right|$$

$$\leq \sup_{t_0 \leq t \leq t_0 + \delta} \int_{t_0}^{t} |g(s,u(s)) - g(s,u(s)) + g(s,u(s)) - g(s,v(s)) + g(s,v(s)) - g(s,v(s))|\, ds$$

$$\leq \sup_{t_0 \leq t \leq t_0 + \delta} \alpha|u(s) - v(s)| + \beta|u(s) - Pu(s)| \;\; + \gamma|v(s) - Pv(s)| \int_{t_0}^{t} ds$$

$$\leq \delta[\,\alpha|u - v| + \beta|u - Pu| \;\; + \gamma|v - Pv|]$$

It follows that if $\delta \leq 1/\,\alpha + \beta + \gamma$ then t has contraction on $[t_0, t_0 + \delta]$ Therefore there is unique solution $u, v\colon [t_0, t_0 + \delta] \to R^n$

Main Results 3

Theorem 3.1:let g(x,y) be a continuous functions of domain g=[a,b] × [c,d] such that g is Lipschitzian with respect to y i.e., there exists k >0 such that

$|P(t,x) - P(t,v)| \leq \alpha\,|x - v| + \beta|x - Px| + \gamma|v - Pv|$for all u,v ∈[c,d] and x ∈[a,b]

Suppose (x_0; y_0)∈int(Dom) g.Then for sufficiently small h>0, there exist a unique solution of the problem (2.1)

<u>Proof</u>: let M = sup{$|g(x,y)|$: x,y ∈ Dom(f) }and choose h >0 such that

C = { y∈ C [$x_0 - h, x_0 + h$]: $|y(x) - y_0|$< Mh }

Then C is closed subset of complex metric space C [$x_0 - h, x_0 + h$]and hence C is complete. Note that P: C → C is contraction mapping.Indeed for x∈ [$x_0 - h, x_0 + h$] and two continuous functions y_1,y_2∈ C we have

$$\|Py_1 - Py_2\| = \left\|\int_{x0}^{x} g(x,y_1) - g(x,y_2)\ dt\right\|$$

$$\leq |x - x_0| \sup_{s\in[x_0 - h, x_0+h]} \alpha\,|y_1(s) - y_2(s)| + \beta|y_1(s) - Py_1(s)| + \gamma|y_2(s) - Py_2\,(s)|$$

$$\leq\ h\alpha\,|y_1 - y_2| + \beta|y_1 - Py_1| + \gamma|y_2 - Py_2|$$

Therefore,P has unique fixed point implying that problem 2.1 has unique fixed point.

CONCLUSION:

We have prolonged the theorems of BoydWong[3] and calming the upper semi-continuity of the functionθ in theorem 2.1.Additional if we addα, β, δ, in Lipschitz condition and becomes modified Lipschitz condition. Using modified Lipschitz condition proved uniqueness of ordinary differential equation.

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REFERENCE:

1. Piyachat Borisut, Konrawut Khammahawong and Poom Kumam, Fixed Point Theory Approach to Existence of Solutions with Differential Equations,
2. Hunter JK, Nachtergaele B. Applied Analysis. River Edge, NJ:World Scientific Publishing Co., Inc.; 2001
3. Boyd, D. W., Wong, J. S. W., Nonlinear contractions. Proc. Am. Math. Soc. 20, 458–464 (1969).
4. Kreyszig, E., Introductory Functional Analysis with Applications. Wiley Classics Library. Wiley, New York (1989).
5. Nalwade V V ,Dolhare U P :Generalization of Banach contraction mapping principle and fixed point theorem by altering distances between the points. Internal national journal of research and analytical reviews, 4(3),19-26.(July-Sept 2017).
6. Khan, M. S., Swaleh, M. and Sessa, S., “Fixed point theorems by altering distances between the points” Bulletin of the Australian Mathematical Society, vol. 30, no. 1, pp. 1-9, 1984.
7. Plaat O. Ordinary Differential Equations. San Francisco: Holden-Day; 1971
8. Hunter JK, Nachtergaele B. Applied Analysis. River Edge, NJ: World Scientific Publishing Co., Inc.; 2001.
9. Banach, S., Sur les operations dans les ensembles abstraitsetleur applications aux equations integrals, Fund. Math. 3, 133–181 (1922).

10. Matkowski, J., Integrable solutions of functional equations. Diss. Math. 127, (1975).