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Relation between Second, Third and Fourth Coefficients for Subclasses of Pascu Classes of Analytic Functions

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ABSTACT: We define two subclasses of the class of Pascu functions. For any real μ, we are interested in determiningthe upper bound of $|a_2a_4 - \mu a_3^2|$ for an analytic function $f(z) = z + a_2z^2 + a_3z^3 + ----$ $(|z| < 1)$ belonging to theseclasses.

MATHEMATICS SUBJECT CLASSIFICATION: 30C45

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1. INTRODUCTION AND DEFINITION:

PRINCIPLES OF SUBORDINATION:Let $f(z)$ and $F(z)$ be two analytic functions in the unit disc $E = \{z : |z| < 1\}$. Then, $f(z)$ is said to be subordinate to $F(z)$ in the unit disc E if there exists an analytic function $w(z)$ in E satisfying the condition $w(0) = 0$, $|w(z)| < 1$ such that $f(z) = F(w(z))$ and we write as $f(z) \prec F(z)$.In particular if $F(z)$ is univalent in D, the above definition is equivalent to $f(0) = F(0)$ and $f(E) \subset F(E)$.

FUNCTIONS WITH POSITIVE REAL PART: Let Ƥ denotes the class of analytic functions of the form

(1.1) $$P(z) = 1 + p_1z + p_2z^2 + ...$$

with $\operatorname{Re} P(z) > 0,\ z \in D$.

Let A denote the class of functions of the form

(1.2) $$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

which are analytic in the open unit disc $D = \{z : |z| < 1\}$.

S is the class of functions of the form (1.2) which are univalent.

The Hankel determinant: ([9],[10])

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be analytic in D. For $q \geq 1,$ the qth Hankel determinant is defined by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} \ldots\ldots & a_{n+q-1} \\ a_{n+1} & a_{n+2} \ldots\ldots & a_{n+q} \\ a_{n+q-1} & a_{n+q} \ldots\ldots. & a_{n+2q-2} \end{vmatrix}.$$

The Hankel determinant was studied by various authors including Hayman[3] and Ch. Pommerenke([13],[14]). For q= 2 and n = 2 , the second Hankel determinant for the analytic function $f(z)$ is defined by

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = \left(a_2 a_4 - a_3^2\right)$$

R_0 represents the class of functions $f(z) \in A$ and satisfying the condition

(1.3) $$\operatorname{Re}\left[\frac{f(z)}{z}\right] > 0,\ z \in D.$$

R_0 is a particular case of the class of close to star function defined by Reade[17]. The class R_0 and its subclasses were vastly studied by several authors including Mac-Gregor[7].

Let R be the class of functions $f(z) \in A$ and satisfying

(1.4) $$\operatorname{Re} f'(z) > 0,\ z \in D.$$

The class R was introduced by Noshiro [11] and Warschawski[18] (known as N-W class) and it was shown by them that R is a class of univalent functions. The class R and its subclasses were investigated by various authors including Goel and the author ([1],[2]).

For $\alpha \geq 0$, $R_1(\alpha)$ and $R_2(\alpha)$ denote the classes of functions in A which satisfy, respectively, the conditions

(1.5) $$\mathrm{Re}\left[(1-\alpha)\frac{f(z)}{z}+\alpha f'(z)\right]>0,\ z\in D$$

and

(1.6) $$\mathrm{Re}\left[f'(z)+\alpha z f''(z)\right]>0,\ z\in D.$$

The classes $R_1(\alpha)$ and $R_2(\alpha)$ were introduced by Pascu [12] and are called Pascu classes of functions. It is obvious that $f(z)\in R_1(\alpha)$ implies that $zf'(z)\in R_2(\alpha)$.

We shall deal with the following classes

(1.7) $$R_1(\alpha;A,B)=\left\{f\in A:\left[(1-\alpha)\frac{f(z)}{z}+\alpha f'(z)\prec\frac{1+Az}{1+Bz},\alpha\geq 0,-1\leq B<A\leq 1,z\in D\right]\right\}$$

and

(1.8) $$R_2(\alpha;A,B)=\left\{f\in A:\left[f'(z)+\alpha z f''(z)\prec\frac{1+Az}{1+Bz},\alpha\geq 0,-1\leq B<A\leq 1,z\in D\right]\right\}.$$

$R_1(\alpha;1,-1)\equiv R_1(\alpha)$ and $R_2(\alpha;1,-1)\equiv R_2(\alpha)$. $R_1(\alpha;A,B)$ is a subclass of $R_1(\alpha)$ and $R_2(\alpha;A,B)$ is a sublass of $R_2(\alpha)$. The classes $R_1(\alpha;A,B)$ and $R_2(\alpha;A,B)$ were studied by the author[8].Througout the paper, we assume that $\alpha\geq 0,-1\leq B<A\leq 1$ and $z\in D$.

1. PRELIMINARY LEMMAS

Lemma 2.1 [15]. Let $P(z)\in \mathcal{P}(z)$, then

$$|p_n|\leq 2\ (n=1,2,3,..)$$

Lemma 2.2 [5]. Let $P(z)\in \mathcal{P}(z)$, then

$$2p_2=p_1^2+\left(4-p_1^2\right)x,$$

$$4p_3 = p_1^3 + 2p_1\left(4 - p_1^2\right)x - p_1\left(4 - p_1^2\right)x^2 + 2\left(4 - p_1^2\right)\left(1 - |x|^2\right)z$$

for some x and z with $|x| \le 1, |z| \le 1$ and $p_1 \in [0,2]$.

2. MAIN RESULTS

Theorem 3.1: Let $f \in R_1(\alpha; A, B)$, then

$$\left|a_2 a_4 - \mu a_3^2\right| \le$$

(3.1) $$\left(\frac{1-B}{A-B}\right)^2\left[\frac{\left\{3(1+2\alpha)^2 - 2\mu(1+\alpha)(1+3\alpha)\right\}^2}{2(1+\alpha)(1+3\alpha)(1+2\alpha)^2\left\{(1+2\alpha)^2 - \mu(1+\alpha)(1+3\alpha)\right\}} - \frac{4\mu}{(1+2\alpha)^2}\right] \text{ if } \mu \le 0;$$

(3.2) $$\left(\frac{1-B}{A-B}\right)^2\left[\frac{\left\{3(1+2\alpha)^2 - 4\mu(1+\alpha)(1+3\alpha)\right\}^2}{2(1+\alpha)(1+3\alpha)(1+2\alpha)^2\left\{(1+2\alpha)^2 - \mu(1+\alpha)(1+3\alpha)\right\}} + \frac{4\mu}{(1+2\alpha)^2}\right]$$
$$\text{if } 0 \le \mu \le \frac{3(1+2\alpha)^2}{4(1+\alpha)(1+3\alpha)};$$

(3.3) $$\left(\frac{1-B}{A-B}\right)^2\left[\frac{4\mu}{(1+2\alpha)^2}\right] \text{ if } \frac{3(1+2\alpha)^2}{4(1+\alpha)(1+3\alpha)} \le \mu \le \frac{3(1+2\alpha)^2}{2(1+\alpha)(1+3\alpha)};$$

(3.4) $$\left(\frac{1-B}{A-B}\right)^2\left[\frac{\left\{2\mu(1+\alpha)(1+3\alpha) - 3(1+2\alpha)^2\right\}^2}{2(1+\alpha)(1+3\alpha)(1+2\alpha)^2\left\{\mu(1+\alpha)(1+3\alpha) - (1+2\alpha)^2\right\}} + \frac{4\mu}{(1+2\alpha)^2}\right]$$
$$\text{if } \mu \ge \frac{3(1+2\alpha)^2}{2(1+\alpha)(1+3\alpha)}.$$

Proof. By definition of subordination,

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) = \frac{1+Aw(z)}{1+Bw(z)},$$

Taking real parts,

$$\operatorname{Re}\left[(1-\alpha)\frac{f(z)}{z} + \alpha f'(z)\right] = \operatorname{Re}\left[\frac{1+Aw(z)}{1+Bw(z)}\right]$$

$$\geq \frac{1-Ar}{1-Br} > \frac{1-A}{1-B} \qquad (|z| = r)$$

which implies that

$$(3.5) \quad 1 + \frac{1-B}{A-B}\left[(1+\alpha)a_2 z + (1+2\alpha)a_3 z^2 + (1+3\alpha)a_4 z^3 + \ldots\right] = P(z) \ .$$

Equating the coefficients in (3.5), we get

$$(3.6) \quad \begin{cases} a_2 = \left(\frac{1-B}{A-B}\right)\frac{p_1}{(1+\alpha)} \\ a_3 = \left(\frac{1-B}{A-B}\right)\frac{p_2}{(1+2\alpha)} \\ a_4 = \left(\frac{1-B}{A-B}\right)\frac{p_3}{(1+3\alpha)} \end{cases}$$

System (3.6) ensures that

$$(3.7) \quad C(\alpha)\left(a_2 a_4 - \mu a_3^2\right) = (1+2\alpha)^2 p_1 (4 p_3) - \mu(1+\alpha)(1+3\alpha)(2 p_2)^2 \ ,$$

$$(3.8) \quad C(\alpha) = 4\left(\frac{A-B}{1-B}\right)^2 (1+\alpha)(1+3\alpha)(1+2\alpha)^2 \ .$$

Using Lemma 2.2 in (3.7), we obtain

$$C(\alpha)\left(a_2 a_4 - \mu a_3^2\right) = (1+2\alpha)^2 p_1\left[p_1^3 + 2p_1\left(4-p_1^2\right)x - p_1\left(4-p_1^2\right)x^2 + 2\left(4-p_1^2\right)\left(1-|x|^2\right)z\right]$$

$$-\mu(1+\alpha)(1+3\alpha)\left[p_1^2 + \left(4-p_1^2\right)x\right]^2$$

for some x and z with $|x| \leq 1$, $|z| \leq 1$. or

$$(3.9) \quad C(\alpha)\left(a_2 a_4 - \mu a_3^2\right) = \left[(1+2\alpha)^2 - \mu(1+\alpha)(1+3\alpha)\right]p_1^4$$

$$+ 2\left[(1+2\alpha)^2 - \mu(1+\alpha)(1+3\alpha)\right]p_1^2\left(4-p_1^2\right)x$$
$$-\left(4-p_1^2\right)\left[\left\{(1+2\alpha)^2 - \mu(1+\alpha)(1+3\alpha)\right\}p_1^2 + 4\mu(1+\alpha)(1+3\alpha)\right]x^2$$
$$+ 2(1+2\alpha)^2 p_1\left(4-p_1^2\right)\left(1-|x|^2\right)z$$

Replacing p_1 by $p \in [0,2]$ and applying triangular inequality to (3.9), we get

$$C(\alpha)\left|a_2a_4-\mu a_3^2\right| \le \begin{cases} \left|(1+2\alpha)^2-\mu(1+\alpha)(1+3\alpha)\right|p^4+2\left|(1+2\alpha)^2-\mu(1+\alpha)(1+3\alpha)\right|p^2\left(4-p^2\right)\delta \\ +\left(4-p^2\right)\left[\ \left|(1+2\alpha)^2-\mu(1+\alpha)(1+3\alpha)\ \right|p^2+|4\mu|(1+\alpha)(1+3\alpha)\right]\delta^2 \\ +2(1+2\alpha)^2\,p\left(4-p^2\right)\left(1-\delta^2\right),\left(\delta=|x|\le 1\right) \end{cases}$$

which can be put in the form

(3.10)
$$C(\alpha)\left|a_2a_4-\mu a_3^2\right| \le \begin{cases} \left[(1+2\alpha)^2-\mu(1+\alpha)(1+3\alpha)\right]p^4+2(1+2\alpha)^2\,p\left(4-p^2\right) \\ +2\left[(1+2\alpha)^2-\mu(1+\alpha)(1+3\alpha)\right]p^2\left(4-p^2\right)\delta \\ +\left(4-p^2\right)\left[\left\{(1+2\alpha)^2-\mu(1+\alpha)(1+3\alpha)\right\}p^2-4\mu(1+\alpha)(1+3\alpha)-2p(1+2\alpha)^2\right]\delta^2 \\ if\mu\le 0; \\ \\ \left[(1+2\alpha)^2-\mu(1+\alpha)(1+3\alpha)\right]p^4+2(1+2\alpha)^2\,p\left(4-p^2\right) \\ +2\left[(1+2\alpha)^2-\mu(1+\alpha)(1+3\alpha)\right]p^2\left(4-p^2\right)\delta \\ +\left(4-p^2\right)\left[\left\{(1+2\alpha)^2-\mu(1+\alpha)(1+3\alpha)\right\}p^2+4\mu(1+\alpha)(1+3\alpha)-2p(1+2\alpha)^2\right]\delta^2 \\ if\,0\le\mu\le\dfrac{(1+2\alpha)^2}{(1+\alpha)(1+3\alpha)}; \\ \\ \left[\mu(1+\alpha)(1+3\alpha)-(1+2\alpha)^2\right]p^4+2(1+2\alpha)^2\,p\left(4-p^2\right) \\ +2\left[\mu(1+\alpha)(1+3\alpha)-(1+2\alpha)^2\right]p^2\left(4-p^2\right)\delta \\ +\left(4-p^2\right)\left[\left\{\mu(1+\alpha)(1+3\alpha)-(1+2\alpha)^2\right\}p^2+4\mu(1+\alpha)(1+3\alpha)-2p(1+2\alpha)^2\right]\delta^2 \\ if\mu\ge\dfrac{(1+2\alpha)^2}{(1+\alpha)(1+3\alpha)} \end{cases}$$

$= F(\delta)$.

$F'(\delta)>0$ and therefore $F(\delta)$ is increasing in [0,1]. $F(\delta)$attains its maximum value at $\delta=1$.

(3.10) reduces to

$$C(\alpha)\left|a_2a_4-\mu a_3^2\right| \le \begin{cases} \left[(1+2\alpha)^2-\mu(1+\alpha)(1+3\alpha)\right]p^4+2\left[(1+2\alpha)^2-\mu(1+\alpha)(1+3\alpha)\right]p^2\left(4-p^2\right) \\ +\left(4-p^2\right)\left[\left\{(1+2\alpha)^2-\mu(1+\alpha)(1+3\alpha)\right\}p^2-4\mu(1+\alpha)(1+3\alpha)\right]if\mu\le 0; \\ \\ \left[(1+2\alpha)^2-\mu(1+\alpha)(1+3\alpha)\right]p^4+2\left[(1+2\alpha)^2-\mu(1+\alpha)(1+3\alpha)\right]p^2\left(4-p^2\right) \\ +\left(4-p^2\right)\left[\left\{(1+2\alpha)^2-\mu(1+\alpha)(1+3\alpha)\right\}p^2+4\mu(1+\alpha)(1+3\alpha)\right]if\,0\le\mu\le\dfrac{(1+2\alpha)^2}{(1+\alpha)(1+3\alpha)}; \\ \\ \left[\mu(1+\alpha)(1+3\alpha)-(1+2\alpha)^2\right]p^4+2\left[\mu(1+\alpha)(1+3\alpha)-(1+2\alpha)^2\right]p^2\left(4-p^2\right) \\ +\left(4-p^2\right)\left[\left\{\mu(1+\alpha)(1+3\alpha)-(1+2\alpha)^2\right\}p^2+4\mu(1+\alpha)(1+3\alpha),if\mu\ge\dfrac{(1+2\alpha)^2}{(1+\alpha)(1+3\alpha)}\right] \end{cases}$$

$= G(p)$ or

(3.11) $C(\alpha)\left|a_2a_4-\mu a_3^2\right| \le \max G(p),$

Case (i) $\mu \le 0$

$$G(p) = -2\left[(1+2\alpha)^2-\mu(1+\alpha)(1+3\alpha)\right]p^4+4\left[3(1+2\alpha)^2-2\mu(1+\alpha)(1+3\alpha)\right]p^2-16(1+\alpha)(1+3\alpha)\mu$$

$G(p)$ is maximum for

$$G'(p) = -8\left[(1+2\alpha)^2-\mu(1+\alpha)(1+3\alpha)\right]p^3+8\left[3(1+2\alpha)^2-2\mu(1+\alpha)(1+3\alpha)\right]p = 0$$

which implies that $p = \sqrt{\dfrac{\left[3(1+2\alpha)^2-2\mu(1+\alpha)(1+3\alpha)\right]}{\left[(1+2\alpha)^2-\mu(1+\alpha)(1+3\alpha)\right]}}.$

Putting the corresponding value of $G(p)$ along with $C(\alpha)$ from (3.8) in (3.11), we get (3.1)

Case (ii) $0 \le \mu \le \dfrac{(1+2\alpha)^2}{(1+\alpha)(1+3\alpha)}$

$$G(p) = -2\left[(1+2\alpha)^2-\mu(1+\alpha)(1+3\alpha)\right]p^4+4\left[3(1+2\alpha)^2-4\mu(1+\alpha)(1+3\alpha)\right]p^2+16(1+\alpha)(1+3\alpha)\mu$$

Sub-case (a) $0 \le \mu \le \dfrac{3(1+2\alpha)^2}{4(1+\alpha)(1+3\alpha)}$

It is easy to see that $G(p)$ is maximum at

$$p = \sqrt{\frac{\left[3(1+2\alpha)^2-4\mu(1+\alpha)(1+3\alpha)\right]}{\left[(1+2\alpha)^2-\mu(1+\alpha)(1+3\alpha)\right]}}.$$

Substituting the corresponding value of $G(p)$ and the value of $C(\alpha)$ in (3.11), (3.2) follows

Sub-case (b) $\dfrac{3(1+2\alpha)^2}{4(1+\alpha)(1+3\alpha)} \le \mu \le \dfrac{(1+2\alpha)^2}{(1+\alpha)(1+3\alpha)}$

$G'(p) < 0$ and $G(p)$ is maximum at p=0

In this sub-case max $G(p) = 16(1+\alpha)(1+3\alpha)\mu$.

Case (iii) $\mu \geq \frac{(1+2\alpha)^2}{(1+\alpha)(1+3\alpha)}$

$$G(p) = -2\left[\mu(1+\alpha)(1+3\alpha) - (1+2\alpha)^2\right]p^4 + 4\left[2\mu(1+\alpha)(1+3\alpha) - 3(1+2\alpha)^2\right]p^2 + 16(1+\alpha)(1+3\alpha)\mu$$

Sub-case (a) $\frac{(1+2\alpha)^2}{(1+\alpha)(1+3\alpha)} \leq \mu \leq \frac{3(1+2\alpha)^2}{2(1+\alpha)(1+3\alpha)}$

$G'(p) < 0$ and maximum $G(p) = G(0) = 16(1+\alpha)(1+3\alpha)\mu$

Combining the cases (ii)-(b) and (iii)-(a) we arrive at (3.3)

Sub-case (b) $\mu \geq \frac{3(1+2\alpha)^2}{2(1+\alpha)(1+3\alpha)}$

A simple calculus shows that $G(p)$ is maximum at $p = \sqrt{\frac{\left[2\mu(1+\alpha)(1+3\alpha) - 3(1+2\alpha)^2\right]}{\left[\mu(1+\alpha)(1+3\alpha) - (1+2\alpha)^2\right]}}$

Substituting the corresponding value of $G(p)$ and the value of $C(\alpha)$ in (3.11), (3.4) follows

Remark 3.1 Put A=1 and B= -1 in the theorem we get the estimates for the class $R_1(\alpha)$.

Taking A=1, B= -1 and $\alpha = 0$ in the theorem we have

Corollary 3.1 If $f \in R_0$, then

$$\left|a_2 a_4 - \mu a_3^2\right| \leq \begin{cases} \frac{(3-2\mu)^2}{2(1-\mu)} - 4\mu, \mu \leq 0; \\ \frac{(3-4\mu)^2}{2(1-\mu)} + 4\mu, 0 \leq \mu \leq \frac{3}{4}; \\ 4\mu, \frac{3}{4} \leq \mu \leq \frac{3}{2}; \\ \frac{(2\mu-3)^2}{2(\mu-1)} + 4\mu, \mu \geq \frac{3}{2}. \end{cases}$$

Letting A=1,B= -1 and $\alpha = 1$ we get

Corollary 3.2 If $f \in R$, then

$$\left|a_2 a_4 - \mu a_3^2\right| \le \begin{cases} \dfrac{(27-16\mu)^2}{144(9-8\mu)} - \dfrac{4\mu}{9}\, \mu \le 0; \\ \dfrac{(27-32\mu)^2}{144(9-8\mu)} + \dfrac{4\mu}{9}, 0 \le \mu \le \dfrac{27}{32}; \\ \dfrac{4\mu}{9}, \dfrac{27}{32} \le \mu \le \dfrac{27}{16};; \\ \dfrac{(16\mu-27)^2}{144(8\mu-9)} + \dfrac{4\mu}{9}, \mu \ge \dfrac{27}{16}. \end{cases}$$

Thisresults was proved by Jantenget al [4]

Theorem 3.2 Let $f \in R_2(\alpha; A, B)$, then

$$\left|a_2 a_4 - \mu a_3^2\right| \le$$

$$\begin{cases} (3.12) \quad \left(\dfrac{1-B}{A-B}\right)^2 \left(\dfrac{\{27(1+2\alpha)^2 - 16\mu(1+\alpha)(1+3\alpha)\}^2}{144(1+\alpha)(1+3\alpha)(1+2\alpha)^2\{9(1+2\alpha)^2 - 8\mu(1+\alpha)(1+3\alpha)\}} - \dfrac{4\mu}{9(1+2\alpha)^2} \right) \\ if \mu \le 0; \\ (3.13) \quad \left(\dfrac{1-B}{A-B}\right)^2 \left[\dfrac{\{27(1+2\alpha)^2 - 32\mu(1+\alpha)(1+3\alpha)\}^2}{144(1+\alpha)(1+3\alpha)(1+2\alpha)^2\{9(1+2\alpha)^2 - 8\mu(1+\alpha)(1+3\alpha)\}} + \dfrac{4\mu}{9(1+2\alpha)^2} \right] \\ if\, 0 \le \mu \le \dfrac{27(1+2\alpha)^2}{32(1+\alpha)(1+3\alpha)}; \\ (3.14) \quad \left(\dfrac{1-B}{A-B}\right)^2 \left[\dfrac{4\mu}{9(1+2\alpha)^2} \right] \\ if\, \dfrac{27(1+2\alpha)^2}{32(1+\alpha)(1+3\alpha)} \le \mu \le \dfrac{27(1+2\alpha)^2}{16(1+\alpha)(1+3\alpha)}; \\ (3.15) \quad \left(\dfrac{1-B}{A-B}\right)^2 \left[\dfrac{\{16\mu(1+\alpha)(1+3\alpha) - 27(1+2\alpha)^2\}^2}{144(1+\alpha)(1+3\alpha)(1+2\alpha)^2\{8\mu(1+\alpha)(1+3\alpha) - 9(1+2\alpha)^2\}} + \dfrac{4\mu}{9(1+2\alpha)^2} \right] \\ if \mu \ge \dfrac{27(1+2\alpha)^2}{16(1+\alpha)(1+3\alpha)}. \end{cases}$$

Proof. We have

$$f'(z) + \alpha z f''(z) = \frac{1 + Aw(z)}{1 + Bw(z)}$$

Taking real parts,

$$\operatorname{Re}\left[f'(z)+\alpha z f''(z)\right]=\operatorname{Re}\left[\frac{1+Aw(z)}{1+Bw(z)}\right]\geq\frac{1-Ar}{1-Br}>\frac{1-A}{1-B}\ (|z|=r)$$

This implies that

(3.16) $$1+\left(\frac{1-B}{A-B}\right)\left[2(1+\alpha)a_2z+3(1+2\alpha)a_3z^2+4(1+3\alpha)a_4z^3+.......\right]=P(z)$$

Identifying the terms in (3.16), we get

(3.17) $$\begin{cases} a_2=\left(\frac{A-B}{1-B}\right)\frac{p_1}{2(1+\alpha)} \\ a_3=\left(\frac{A-B}{1-B}\right)\frac{p_2}{3(1+2\alpha)} \\ a_4=\left(\frac{A-B}{1-B}\right)\frac{p_3}{4(1+3\alpha)} \end{cases}$$

System (3.17) yields

(3.18) $$C(\alpha)\left(a_2a_4-\mu a_3^2\right)=9(1+2\alpha)^2p_1(4p_3)-8\mu(1+\alpha)(1+3\alpha)(2p_2)^2,$$

(3.19) $$C(\alpha)=\left(\frac{A-B}{1-B}\right)^2\left[288(1+\alpha)(1+3\alpha)(1+2\alpha)^2\right]$$

By Lemma 2.2, (3.19) can be written as

$$C(\alpha)\left(a_2a_4-\mu a_4^2\right)=9(1+2\alpha)^2p_1\left[p_1^3+2p_1\left(4-p_1^2\right)x-p_1\left(4-p_1^2\right)x^2+2\left(4-p_1^2\right)\left(1-|x|^2\right)z\right]$$

$$-8\mu(1+\alpha)(1+3\alpha)\left[p_1^2+\left(4-p_1^2\right)x\right]^2$$

for some x and z with $|x|\leq 1, |z|\leq 1.$

or

(3.20) $$C(\alpha)\left(a_2a_4-\mu a_3^2\right)=\left[9(1+2\alpha)^2-8\mu(1+\alpha)(1+3\alpha)\right]p_1^4$$

$$+2\left[9(1+2\alpha)^2-8\mu(1+\alpha)(1+3\alpha)\right]p_1^2\left(4-p_1^2\right)x$$

$$-\left(4-p_1^2\right)\left[\left\{9(1+2\alpha)^2-8\mu(1+\alpha)(1+3\alpha)+32\mu(1+\alpha)(1+3\alpha)\right\}\right]x^2$$

$$+18(1+2\alpha)^2p_1\left(4-p_1^2\right)\left(1-|x|^2\right)z\,.$$

Replacing p_1 by $p\in[0,2]$ and applying triangular inequality to (3.20), we get

$$C(\alpha)\left|a_2a_4-\mu a_3^2\right|\le\left|9(1+2\alpha)^2-8\mu(1+\alpha)(1+3\alpha)\right|p^4$$

$$2\left|9(1+2\alpha)^2-8\mu(1+\alpha)(1+3\alpha)\right|p^2\left(4-p^2\right)\delta$$

$$+\left(4-p^2\right)\left[\ \left|9(1+2\alpha)^2-8\mu(1+\alpha)(1+3\alpha)\ \right|p^2+|32\mu|(1+\alpha)(1+3\alpha)\right]\delta^2$$

$$+18(1+2\alpha)^2p\left(4-p^2\right)\left(1-|\delta|^2\right).\ (\delta=|x|\le1)$$

which can be put in the form

$$C(\alpha)\left|a_2a_4-\mu a_3^2\right|\le\begin{cases}\left[9(1+2\alpha)^2-8\mu(1+\alpha)(1+3\alpha)\right]p^4+18(1+2\alpha)^2p\left(4-p^2\right)\\+2\left[9(1+2\alpha)^2-8\mu(1+\alpha)(1+3\alpha)\right]p^2\left(4-p^2\right)\delta\\+\left(4-p^2\right)\begin{bmatrix}\left\{9(1+2\alpha)^2-8\mu(1+\alpha)(1+3\alpha)\right\}p^2-32\mu(1+\alpha)(1+3\alpha)\\-18(1+2\alpha)^2p\end{bmatrix}\delta^2 if \mu\le0;\\ \\ \left[9(1+2\alpha)^2-8\mu(1+\alpha)(1+3\alpha)\right]p^4+18(1+2\alpha)^2p\left(4-p^2\right)\\+2\left[9(1+2\alpha)^2-8\mu(1+\alpha)(1+3\alpha)\right]p^2\left(4-p^2\right)\delta\\\left(4-p^2\right)\begin{bmatrix}\left\{9(1+2\alpha)^2-8\mu(1+\alpha)(1+3\alpha)\right\}p^2+32\mu(1+\alpha)(1+3\alpha)\\-18(1+2\alpha)^2p\end{bmatrix}\delta^2\\ if\,0\le\mu\le\dfrac{9(1+2\alpha)^2}{8(1+\alpha)(1+3\alpha)};\\ \\ \left[8\mu(1+\alpha)(1+3\alpha)-9(1+2\alpha)^2\right]p^4+18(1+2\alpha)^2p\left(4-p^2\right)\\+2\left[8\mu(1+\alpha)(1+3\alpha)-9(1+2\alpha)^2\right]p^2\left(4-p^2\right)\delta\\\left(4-p^2\right)\begin{bmatrix}\left\{8\mu(1+\alpha)(1+3\alpha)-9(1+2\alpha)^2\right\}p^2+32\mu(1+\alpha)(1+3\alpha)\\-18(1+2\alpha)^2p\end{bmatrix}\delta^2\\ if\mu\ge\dfrac{9(1+2\alpha)^2}{8(1+\alpha)(1+3\alpha)}.\end{cases}$$

$$= F(\delta)$$

$F'(\delta) > 0$ which means that $F(\delta)$ is increasing in [0 , 1] and $F(\delta)$ attains maximum value at $\delta = 1$

(3.21) reduces to

$$C(\alpha)\left|\left(a_2a_4 - \mu a_3^2\right)\right| \leq \begin{cases} \left[9(1+2\alpha)^2 - 8\mu(1+\alpha)(1+3\alpha)\right]p^4 + 2\left[9(1+2\alpha)^2 - 8\mu(1+\alpha)(1+3\alpha)\right]p^2\left(4-p^2\right) \\ + \left(4-p^2\right)\left[\left\{9(1+2\alpha)^2 - 8\mu(1+\alpha)(1+3\alpha)\right\}p^2 - 32\mu(1+\alpha)(1+3\alpha)\right], if \mu \leq 0; \\ \\ \left[9(1+2\alpha)^2 - 8\mu(1+\alpha)(1+3\alpha)\right]p^4 + 2\left[9(1+2\alpha)^2 - 8\mu(1+\alpha)(1+3\alpha)p^2\left(4-p^2\right)\right] \\ + (4-p^2)\left[\left\{9(1+2\alpha)^2 - 8\mu(1+\alpha)(1+3\alpha)\right\}p^2 + 32\mu(1+\alpha)(1+3\alpha)\right] \\ if\, 0 \leq \mu \leq \dfrac{9(1+2\alpha)^2}{8(1+\alpha)(1+3\alpha)}; \\ \\ \left[8\mu(1+\alpha)(1+3\alpha) - 9(1+2\alpha)^2\right]p^4 + 2\left[8\mu(1+\alpha)(1+3\alpha) - 9(1+2\alpha)^2\right]p^2\left(4-p^2\right) \\ + \left(4-p^2\right)\left[\left\{8\mu(1+\alpha)(1+3\alpha) - 9(1+2\alpha)^2\right\}p^2 + 32\mu(1+\alpha)(1+3\alpha)\right] \\ if \mu \geq \dfrac{9(1+2\alpha)^2}{8(1+\alpha)(1+3\alpha)}. \end{cases}$$

Or

(3.22) $C(\alpha)\left|\left(a_2a_4 - \mu a_3^2\right)\right| \leq G(p).$

Case (i) $\mu \leq 0$

$$G(p) = -2\left[9(1+2\alpha)^2 - 8\mu(1+\alpha)(1+3\alpha)\right]p^4 + 4\left[27(1+2\alpha)^2 - 16\mu(1+\alpha)(1+3\alpha)\right]p^2$$
$$+128(1+\alpha)(1+3\alpha)\mu\ .$$

$G(p)$ is maximum for

$$G'(p) = -8\left[9(1+2\alpha)^2 - 8\mu(1+\alpha)(1+3\alpha)\right]p^3 + 8\left[27(1+2\alpha)^2 - 16\mu(1+\alpha)(1+3\alpha)\right]p = 0$$

which gives

$$p=\sqrt{\frac{27(1+2\alpha)^2-16\mu(1+\alpha)(1+3\alpha)}{9(1+2\alpha)^2-8\mu(1+\alpha)(1+3\alpha)}}\ .$$

Putting the corresponding value of $G(p)$ along with the value of $C(\alpha)$ from (3.19) in (3.22), we get (3.12).

Case (ii) $0\le\mu\le\frac{9(1+2\alpha)^2}{8(1+\alpha)(1+3\alpha)}$

$$G(p)=-2\left[9(1+2\alpha)^2-8\mu(1+\alpha)(1+3\alpha)\right]p^4+4\left[27(1+2\alpha)^2-32\mu(1+\alpha)(1+3\alpha)\right]p^2$$
$$+128(1+\alpha)(1+3\alpha)\mu$$

Sub-case (a) $0\le\mu\le\frac{27(1+2\alpha)^2}{32(1+\alpha)(1+3\alpha)}$

An elementary calculus shows that $G(p)$ is maximum at

$$p=\sqrt{\frac{27(1+2\alpha)^2-32\mu(1+\alpha)(1+3\alpha)}{9(1+2\alpha)^2-8\mu(1+\alpha)(1+3\alpha)}}\ .$$

With the corresponding value of $G(p)$ along with the value of $C(\alpha)$ in (3.22) , we arrive at (3.13).

Sub-case (b) $\frac{27(1+2\alpha)^2}{32(1+\alpha)(1+3\alpha)}\le\mu\le\frac{9(1+2\alpha)^2}{8(1+\alpha)(1+3\alpha)}$

$G'(p)<0$ and $G(p)$ is maximum at $p=0$.

$$\max G(p)=G(0)=128(1+\alpha)(1+3\alpha)\mu .$$

Case (iii) $\mu\ge\frac{9(1+2\alpha)^2}{8(1+\alpha)(1+3\alpha)}$

$$G(p)=-2\left[8\mu(1+\alpha)(1+3\alpha)-9(1+2\alpha)^2\right]p^4+4\left[16\mu(1+\alpha)(1+3\alpha)-27(1+2\alpha)^2\right]p^2$$
$$+128(1+\alpha)(1+3\alpha)\mu .$$

Sub-case(a) $\frac{9(1+2\alpha)^2}{8(1+\alpha)(1+3\alpha)} \le \mu \le \frac{27(1+2\alpha)^2}{16(1+\alpha)(1+3\alpha)}$

$G'(p) < 0$ and Max $G(p) = G(0) = 128(1+\alpha)(1+3\alpha)\mu$.

Combining the cases (ii-b) and (iii-a), (3.14) follows.

Sub-case (b) $\mu \ge \frac{27(1+2\alpha)^2}{16(1+\alpha)(1+3\alpha)}$

An easy calculation shows that $G(p)$ is maximum at $p = \sqrt{\frac{16\mu(1+\alpha)(1+3\alpha) - 27(1+2\alpha)^2}{8\mu(1+\alpha)(1+3\alpha) - 9(1+2\alpha)^2}}$.

Substituting the corresponding value of $G(p)$ along with the value of $C(\alpha)$ in (3.22), we obtain (3.15).

Remark 3.2 Putting A=1 and B= -1 in the theorem we get the estimates for the class $R_2(\alpha)$.

Remark 3.3 Letting $A = 1, B = -1$ and $\alpha = 0$ in the theorem , corollary 3.2 follows.

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