

**Mathematical Modeling of the dynamics of Prey-Predator with Scavenger in a closed habitat**

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**Abstract**

In this paper we developed a mathematical model which describes the dynamics of prey-predator interaction with scavenger. The model based on Holling type II functional response. Here we tried to develop model using system of non linear differential equation. We solved the equilibrium points and their existence. The positivity and of the solution of the model are also determined. Conditions for local and global stability analysis are studied both analytically and numerically. The study also addresses the effect of extinction of a population and mechanism that three species coexist. As a result the mechanism that three species become coexist if there is large number of prey population compute with small number of predator and average number of scavenger population. The scavenger species also has a great role in stabilizing as well as for coexistence of three species. Numerical simulations are carried out to illustrate the analytical findings. Finally the biological implication of analytical and numerical are discussed critically

**Key words:** Lyapunov function, mathematical model, prey-predator-scavenger, Stability Analysis

**Introduction**

The phenomena systems of phenomena interacting with in a given system are governed by mathematical equation called mathematical model. Environmental ecology is one of field in which mathematical model is applied to quantify the interacting species. The one of the oldest model which quantifies the interacting species in ecology is called prey-predator model. This known model is first time developed independently by Vito Voltera and Alfrid Lotka in 1926(**Chauvet, E.,2002**).

This model consists system of non-linear differential equation. These system of non-linear ordinary differential equation:

$$\begin{cases} \frac{dx}{dt} = x(a - by) \\ \frac{dy}{dt} = y(-c + dx) \end{cases} \text{----- (1)}$$

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Where  $y(t)$  and  $x(t)$  represent, respectively, the predator and prey population as functions. of time. The parameters  $a, b, c, d > 0$ , are interpreted as follows(**Boyce W.E. & DiPrima**

**R.C.,(1977):**

- a denotes the natural growth rate of the prey in the absence of predators,
- b denotes the effect of predation on the prey,
- c denotes the death rate of the predator due to lack of food,
- d denotes the efficiency and propagation rate of the predator in the presence of prey

Without loss of generality, taking parameters  $a = 1, b = 1$  and  $d = 1$ , with fixed point  $(c, 1)$  all solutions in first quadrant are positive. In this equation, predator population is directly proportional to prey population. That is, if there is more prey population, the predator get more food and increases in number. However, when the predator population increases more and more, the prey population in the closed habitat will decreases and eventually, die out. When predators population is high there is scarce food, prey. This increase the competition among predator population and they eventually, decrease due to shortage of food, since prey is the only source of predators. This trend continues as time goes on, implying a stable coexistence of the two populations.

The modified two dimensional Lotka-Volterra is model of logistic growth of two species including the factor called carrying capacity. In this logistic model, two population make oscillating curve, but eventually, amplitude oscillation decreases until die out. Predator-prey model also uses a nonlinear of equations that includes logistic growth of two species, a carrying capacity of the prey, and a predatory factor. The modified Lotka-Volterra predator-prey model is given by

$$\begin{cases} \frac{dx}{dt} = x(1 - bx - y) \\ \frac{dy}{dt} = y(-c + x) \end{cases} \text{----- (2)}$$

Where  $b$  is the carrying capacity of the prey and  $c$  is the death rate of the predator. In the modified model, we find that the populations start as a pair of oscillating curves, but over time the amplitude of the oscillations decrease with each period of time until the curves attain out. This implies that the populations experienced a stable coexistence that saturates after sometime such that the populations will remain constant.

The Lotka-Volterra model indeed may be the simplest possible predator-prey model. It has been criticized as being unrealistic mainly for its structural instability and the assumption of the unlimited growth of the prey population  $x(t)$  in the absence of a predator. Nevertheless, it is a useful tool containing the basic properties of the real predator-prey dynamics, and serves as a robust basis from which it is possible to develop more sophisticated and realistic models.

The model (2) can be naturally generalized for the multi-species case. The generalization of the Lotka-Volterra model (1), for the multi-species case retains the basic features of real ecological systems and, allows us to obtain valuable results that are easy to be interpreted.

For the three-species predator-prey interaction two possibilities arise the two prey-one predator systems

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$$\begin{cases} \frac{dx_1}{dt} = x_1(a_1 - b_1y) \\ \frac{dx_2}{dt} = x_2(a_1 - b_2y) \\ \frac{dy}{dt} = y(-c + d_1x_1 + d_2x_2) \end{cases} \text{----- (3)}$$

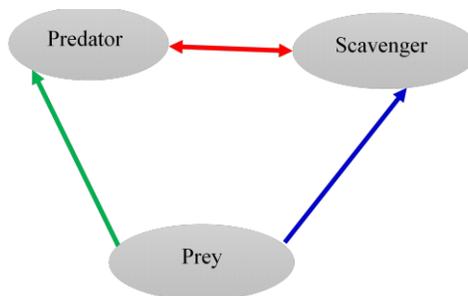
Here  $x_1, x_2$  are prey population and  $y(t)$  is predator population. The interaction of kiwi-rabbits-stoats is an example. The populations of both species are far from reaching their media capacity as a result of predation and artificial controlling of the rabbit population. Furthermore, both the kiwi and the rabbits have the same predators: stoats, cats, minks, or any others. Hence, the kiwi-rabbit-stoat interaction can be adequately described by the equations (3).

And two predator-one prey systems,

$$\begin{cases} \frac{dx}{dt} = x(a_1 - b_1y_1 - b_2y_2) \\ \frac{dy_1}{dt} = y_1(-c_1 + d_1x) \\ \frac{dy_2}{dt} = y_2(-c_2 + d_1x_1 + d_2x) \end{cases} \text{----- (4)}$$

Where as  $y_1, y_2$  are predator population and  $x(t)$  is prey population. For example, lion-tiger-antelope, describe the situation when two predator species depend on a common prey and furthermore, these two predator species do not interact directly, they do not get directly, do not predate one the other, and do not depend one on the other as a food source. Such situations are not unusual, and can arise in many different cases.

We consider a three species, the prey, predator and scavenger where the scavenger is a predator of the prey and scavenges the carcasses of the predator. There is the case where the scavenger has no negative effects on the population that it scavenges. Possible A triple of such species are hyena/lion/antelope, where the hyena scavenges lion carcasses and preys upon antelope. But, in our study the scavenger affect the population it scavenges and also eaten by predator. An example of such a triple can be a lion, zebra and hyena, where the lion is considered as predator, zebra the prey and the hyena represents scavenger. To understand more let us see the diagram below.



**Figure 1: Schematic diagram for the dynamics of the prey, predator and scavenger in an ecosystem**

As we observe from the above diagram both species predator and scavenger use a variety of different resources from two trophic level, so we called as generalist Interaction among these three species can be observed by learning the population growth investigate these growth we develop

the mathematical model which represent the dynamics of the prey, predator and scavenger. In the model we use functional response Holling type II to make the system more real. Then we determine equilibrium points, positivity and boundedness of the solution and also both local as well as global stability of the system.

## 2. Model Formulation

We consider the following predator–prey model with scavenger interaction.

### a) Assumptions and Parameters of the Model

The following assumptions are made in order to construct the model:

- ❖ The prey will grow logistically in the absence of predators and scavenger population. The logistic growth model is illustrated by the term  $AX(1 - \frac{B}{A}X)$
- ❖ the rate of predation upon the prey is proportional to the encounters of predators and prey or the effect on prey population due to interaction with some of predator populations. This assumption is represented by the term  $\frac{BXY}{A_1+X}$
- ❖ the rate of predation upon the prey is proportional to the encounters of scavenger and prey or interaction between prey and scavenger populations. This assumption is represented by the term  $\frac{DXY}{A_1+X}$
- ❖ Predators will die out exponentially in the absence of prey and scavengers or natural death rates of predator populations. This assumption is represented by the term  $EY$ .
- ❖ the predator population increases, due to predation upon prey or benefit of predator population from prey population. The terms  $\frac{FXY}{A_1+X}$  is represents this assumption.
- ❖ the predator population increases, due to predation upon scavenger that is there is a predator population which eat scavenger population. The terms  $\frac{GXY}{A_1+X}$  is represents this assumption.
- ❖ without predators and prey the scavengers will also almost goes to extinct. The terms that represent this assumption are  $HZ$  and  $HZ^2$
- ❖ the term  $HZ$  represent natural death rate of the scavenger and  $HZ^2$  ensures that the interaction within scavenger species itself for the same resource.
- ❖ The scavenger population benefits from prey and predator that die naturally. The terms representing this assumptions are  $\frac{JXY}{A_1+X}$  and  $\frac{KXY}{A_1+X}$  respectively.
- ❖ predators prey and scavengers will come across each other randomly in the environment.
- ❖ The populations live in closed environment

### Parameters of the model

**Table 1: parameters of the model**

Parameters	Interpretation
A	natural growth rate of X
A <sub>1</sub>	half saturation constants for Y and Z
A <sub>2</sub>	half saturation constants for Z
A <sub>3</sub>	half saturation constants for Y

B	interaction between prey population itself
C	effect on x due to predation of Y
D	rate changes on the X population in due to presence of S
E	natural death rate of Y
F	benefit to Y from X
G	benefit to Y from Z
H	natural death rate of Z
I	interaction between S population itself
J	benefit to Z from X
K	benefit to Z from X

### Description of Model

In this model, we considered three species prey, predator and scavenger in modified three species prey-predator model. In this modified three species model, the first species called prey is act as the food for both predator and scavenger. And the second species called predator, feed only on prey and the third species called scavenger is feed on both prey and predator. As result, scavenger population affect the prey population and indirectly reduces the number of predator due to it decreases the prey population which is food for predator.

Depending on these assumptions the model is developed using system of non-linear differential equation:

$$\left\{ \begin{array}{l} \frac{dX}{dt} = AX - BX^2 - \frac{CXY}{A_1+X} - \frac{DXY}{A_1+X} \\ \frac{dY}{dt} = \frac{FXY}{A_1+X} + \frac{GXY}{A_1+Z} - EY \\ \frac{dZ}{dt} = \frac{JXY}{A_1+X} - \frac{KXY}{A_1+Y} - HZ - HZ^2 \end{array} \right. \text{----- (6)}$$

Where X(t), Y(t), Z(t) are the prey, the predator and scavenger populations respectively. All the parameters in the equation (6) are positive.

### Dimensionless form of the model

A model can be transformed into a dimensionless form. That is rewriting the system in terms of dimensionless quantities. One of the advantages of a system in the dimensionless form is that the number of parameters is reduced to a minimum and it makes the analysis easier. Furthermore parameters can be better compared with each other, in terms of small and large and thus one gets more insight into the system. It is also possible to make a comparison between different systems (Murray J.D, 2002). The dynamics of the new model will be the same as in the original system. The equation (6) can be rewritten as:

$$\begin{cases} \frac{dX}{dt} = AX(1 - \frac{B}{A}X) - \frac{CXY}{A_1(1+\frac{X}{A_1})} - \frac{DXY}{A_1(1+\frac{X}{A_1})} \\ \frac{dY}{dt} = -EY + \frac{FXY}{A_1(1+\frac{X}{A_1})} + \frac{GXY}{A_1(1+\frac{Z}{A_1})} \\ \frac{dZ}{dt} = -HZ - HZ^2 + \frac{GXY}{A_1(1+\frac{X}{A_1})} + \frac{KXY}{A_1(1+\frac{X}{A_1})} \end{cases} \dots\dots\dots (7)$$

Letting,  $\tau = At, x = \frac{X}{A_1}, y = \frac{Y}{A_2}, z = \frac{Z}{A_3}$  and substituting (7), dimensionless form of system becomes

$$\begin{cases} \frac{dx}{dt} = x(1 - \alpha_1x) - \frac{\beta_1xy}{(1+x)} - \frac{\theta_1xz}{(1+x)} = f(x, y, z) \\ \frac{dy}{dt} = \frac{\alpha_2xy}{(1+x)} + \frac{\beta_2xy}{(1+z)} - \theta_2y = g(x, y, z) \\ \frac{dz}{dt} = \frac{\alpha_3xz}{(1+x)} + \frac{\beta_3xz}{(1+y)} - \theta_3z - \theta_4z^2 = h(x, y, z) \end{cases} \dots\dots\dots (8)$$

Where  $\alpha_1 = \frac{A_1 B}{A}, \beta_1 = \frac{A_2 C}{AA_1}, \theta_1 = \frac{A_3 D}{AA_1}, \alpha_2 = \frac{F}{A}, \beta_2 = \frac{G}{A}, \theta_2 = \frac{E}{A}, \alpha_3 = \frac{J}{A}, \beta_3 = \frac{K}{A}, \theta_3 = \frac{H}{A}, \theta_4 = \frac{IA_3}{A}$  and  $\tau$  is substituted for  $t$  for simplicity.

**i) Positivity of solution**

Theorem: All solutions of the system (8) are positive

Proof: the equation (8) can be written as

$$x(t) = x(0) \exp\left(\int_0^t \left(1 - \alpha_1x(s) - \frac{\beta_1y(s)}{(1+x(s))} - \frac{\theta_1y(s)}{(1+x(s))}\right) ds\right)$$

$$y(t) = y(0) \exp\left(\int_0^t \left(\frac{\alpha_2x(s)}{(1+x(s))} + \frac{\beta_2x(s)}{(1+z(s))} - \theta_2\right) ds\right)$$

$$z(t) = z(0) \exp\left(\int_0^t \left(\frac{\alpha_3x(s)}{(1+x(s))} + \frac{\beta_3x(s)}{(1+x(s))} - \theta_3 - \theta_4z(s)\right) ds\right)$$

For  $x(0) > 0, y(0) > 0, z(0) > 0$  all solutions remain within the first quadrant of the x-y-z plane starting from an interior point of it.

Hence,  $\mathbb{R}_+^3 = \{(x, y, z): x, y, z \geq 0\}$  is invariant set

**ii) Boundedness of Solution**

Theorem: All solution of (8) are uniformly bounded if the initial conditions  $x(0), y(0), z(0) > 0$ .

Proof: from first equation (8)

$$\frac{dx}{dt} = x(1 - \alpha_1 x) - \frac{\beta_1 xy}{(1+x)} - \frac{\theta_1 xz}{(1+x)} \leq x(1 - \alpha_1 x) \Rightarrow x(t) \leq \frac{1}{\frac{e^{-t}}{x_0} + \alpha_1} \quad \text{For all } t \geq 0, \text{ which}$$

implies  $x(t) < 1$  for sufficiently large  $t$ .

Let  $x(t), y(t), z(t)$  be any positive solution of (8), with positive initial conditions and define that

$$w = \alpha_3 \alpha_2 x(t) + \beta_1 \alpha_3 y(t) + \theta_1 \alpha_2 z(t) \quad \text{Then } \frac{dw}{dt} = \alpha_3 \alpha_2 \frac{dx}{dt} + \beta_1 \alpha_3 \frac{dy}{dt} + \theta_1 \alpha_2 \frac{dz}{dt}$$

$$\text{Therefore } \frac{dw}{dt} = \alpha_3 \alpha_2 (x(1 - \alpha_1 x)) + \beta_1 \alpha_3 \left( \frac{\beta_2 z_0 y}{(1+z_0)} - \theta_2 y \right) + \theta_1 \alpha_2 \left( \frac{\beta_3 y_0 z}{(1+y_0)} - \theta_3 z - \theta_4 z^2 \right)$$

Now choosing,  $\alpha_1 = 1, \beta_2 = \frac{1+z_0}{z_0}, \beta_3 = \frac{1+y_0}{y_0}$  then

$$\frac{dw}{dt} \leq \alpha_3 \alpha_2 (x(1 - x)) + \beta_1 \alpha_3 y + \theta_1 \alpha_2 z \quad \text{Moreover, } \frac{dw}{dt} \leq \alpha_3 \alpha_2 - \gamma(x + y + z)$$

Where  $\gamma = \min \{ \alpha_3 \alpha_2, \beta_1 \alpha_3, \theta_1 \alpha_2 \}$  this implies that  $\frac{dw}{dt} + \gamma w \leq \alpha_3 \alpha_2$  now using method first order ordinary differential equation and applying inequality, we get

$$0 < w < \frac{\alpha_3 \alpha_2 (1 - e^{-\gamma t})}{\gamma} + w(x_0, y_0, z_0) e^{-\gamma t} \quad \text{for } t \rightarrow \infty, w \rightarrow \frac{\alpha_3 \alpha_2}{\gamma}$$

Therefore there exist  $\eta_1, \eta_2, \eta_3 > 0$  such that  $\mathbb{R}_+^3 = \{(x, y, z) : 0 \leq x \leq \eta_1, 0 \leq y \leq \eta_2, 0 \leq z \leq \eta_3\}$  hence, our system is bounded.

### iii) Equilibrium points and their existence

The equilibriums points of the system (8) can be found that setting all the equation to zero and solving the system for  $x(t), y(t)$  and  $z(t)$ . To find equilibriums points of the system, we solve the following system equations simultaneously

$$\begin{cases} \frac{dx}{dt} = x(1 - \alpha_1 x) - \frac{\beta_1 xy}{(1+x)} - \frac{\theta_1 xz}{(1+x)} = f(x, y, z) = 0 \\ \frac{dy}{dt} = \frac{\alpha_2 xy}{(1+x)} + \frac{\beta_2 xy}{(1+z)} - \theta_2 y = g(x, y, z) = 0 \\ \frac{dz}{dt} = \frac{\alpha_3 xz}{(1+x)} + \frac{\beta_3 xz}{(1+y)} - \theta_3 z - \theta_4 z^2 = h(x, y, z) = 0 \end{cases} \quad \text{----- (9)}$$

That is equivalent to solving eight equilibriums points of system (9), namely,

$$E_0(0,0,0), E_1(x^*, 0,0), E_2(0, y^*, 0), E_3(0,0, z^*), E_4(x^*, y^*, 0), E_5(x^*, 0, z^*), E_6(0, y^*, z^*), E_7(x^*, y^*, z^*)$$

With

$$\{E_0(0,0,0), E_1\left(\frac{1}{\alpha}, 0,0\right), E_2(0,0,0), E_3\left(0,0, \frac{-\theta_3}{\theta_4}\right), E_4\left(\frac{\theta_2}{\alpha_2 - \theta_2}, \frac{\alpha_2((\alpha_2 - \theta_2) - \alpha_1 \theta_2)}{\beta_1(\alpha_2 - \theta_2)^2}, 0\right), E_5(x^*, 0, z^*), E_6\left(0, \frac{\theta(\beta - \theta) + \theta\theta}{(\beta_3 - \theta_3)(\beta_2 - \theta_2) - \theta_2\theta_4}, \frac{\theta_2}{\beta_2 - \theta_2}\right), E_7(x^*, y^*, z^*)\}$$

Where  $E_5(x^*, 0, z^*)$  and  $E_7(x^*, y^*, z^*)$  determined using numerically.

**Local stability analysis of equilibrium points**

The local stability of each equilibrium points are studied using Jacobeans matrix and finding Eigen value at each equilibrium points. We continue analysis of Jacobeans matrix J of three dimensional systems by

$$J = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} & \frac{\partial h}{\partial z} \end{bmatrix}, \quad J(x, y, z) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} & \frac{\partial h}{\partial z} \end{bmatrix}$$

$$J(x, y, z) = \begin{bmatrix} 1 - 2\alpha x - \frac{y\beta_1}{(1+x)^2} - \frac{y\theta_1}{(1+x)^2} & -\frac{x\beta_1}{1+x} & -\frac{x\theta_1}{1+x} \\ \frac{y\alpha_2}{(1+x)^2} & -\frac{x\alpha_2}{1+x} - \frac{z\beta_2}{1+z} - \theta_2 & \frac{z\beta_3}{(1+z)^2} \\ \frac{z\alpha_3}{(1+x)^2} & \frac{z\beta_3}{(1+y)^2} & \frac{x\alpha_3}{1+x} - \frac{y\beta_3}{1+y} - 2\theta_4 z - \theta_3 \end{bmatrix} \text{-----(10)}$$

**i) Local stability of  $E_0(0, 0, 0)$**

The Jacobian of matrix at  $E_0(0,0,0)$  of (10) is

$$J(0,0,0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\theta_2 & 0 \\ 0 & 0 & -\theta_3 \end{bmatrix}, \text{ since } J(0,0,0) \text{ is diagonal matrix, with eigenvalues:}$$

1,  $-\theta_2$ , and  $-\theta_3$ . this shows that our system is unstable at  $(0,0,0)$  because the sign of eigen values are different.

**ii) Local stability at  $E_1\left(\frac{1}{\alpha}, 0, 0\right)$**

$$J\left(\frac{1}{\alpha}, 0, 0\right) = \begin{bmatrix} -1 & -\frac{\beta_1}{1+\alpha_1} & -\frac{\theta_1}{1+\alpha_1} \\ 0 & \frac{\alpha_2}{1+\alpha_1} - \theta_2 & 0 \\ 0 & 0 & \frac{\alpha_3}{1+\alpha_1} - \theta_3 \end{bmatrix}, \text{ since } J\left(\frac{1}{\alpha}, 0, 0\right) \text{ is upper triangular matrix, the}$$

eigen values are:  $-1$ ,  $\frac{\alpha_2}{1+\alpha_1} - \theta_2$  and  $\frac{\alpha_3}{1+\alpha_1} - \theta_3$ . Now the system to be stable at  $E_1\left(\frac{1}{\alpha}, 0, 0\right)$ , the two conditions must satisfy:

- a)  $\frac{\alpha_2}{1+\alpha_1} < \theta_2$ ,
- b)  $\frac{\alpha_3}{1+\alpha_1} < \theta_3$ .

If these two conditions holds true then our system is local asymptotically stable at  $E_1\left(\frac{1}{\alpha}, 0, 0\right)$ .

iii) **Local stability at  $E_2(0, 0, 0)$**

Which is the same as i) above?

iv) **Local stability at  $E_3(0, 0, \frac{-\theta_3}{\theta_4})$ ?**

v) **Local stability at  $E_4(\frac{\theta_2}{\alpha_2 - \theta_2}, \frac{\alpha_2((\alpha_2 - \theta_2) - \alpha_1 \theta_2)}{\beta_1(\alpha_2 - \theta_2)^2}, 0)$**

$$J(E_4) = \begin{bmatrix} \frac{\alpha_2 \theta_2 - \alpha_1 \alpha_2 \theta_2 - \alpha_1 \theta_2^2 - \theta_2^2}{\alpha_2(\alpha_2 - \theta_2)} & -\frac{\beta_1 \theta_2}{\alpha_2} & -\frac{\theta_1 \theta_2}{\alpha_2} \\ \frac{\alpha_2 - \theta_2 - \beta_1 \theta_2}{\beta_1} & 0 & 0 \\ 0 & 0 & \frac{\alpha_2 \beta_3(\alpha_2 - \theta_2) - \alpha_1 \alpha_2 \theta_2 \beta_3 - \alpha_1 \theta_2^2 - \theta_2^2}{\beta_1(\alpha_2 - \theta_2)^2 + \alpha_2(\alpha_2 - \theta_2) - \alpha_1 \alpha_2 \theta_2} + \frac{\theta_2 \alpha_3}{\alpha_2} - \theta_3 \end{bmatrix}$$

The characteristic equation can be

$$\lambda^3 - (a_{11} + a_{33})\lambda^2 + (a_{11} a_{33} - a_{12} a_{21})\lambda + a_{12} a_{21} a_{33} = 0 \quad \text{Where, } a_{11} = \frac{\alpha_2 \theta_2 - \alpha_1 \alpha_2 \theta_2 - \alpha_1 \theta_2^2 - \theta_2^2}{\alpha_2(\alpha_2 - \theta_2)}, a_{12} = -\frac{\beta_1 \theta_2}{\alpha_2}, a_{13} = -\frac{\theta_1 \theta_2}{\alpha_2}, a_{33} = \frac{\alpha_2 \beta_3(\alpha_2 - \theta_2) - \alpha_1 \alpha_2 \theta_2 \beta_3 - \alpha_1 \theta_2^2 - \theta_2^2}{\beta_1(\alpha_2 - \theta_2)^2 + \alpha_2(\alpha_2 - \theta_2) - \alpha_1 \alpha_2 \theta_2} + \frac{\theta_2 \alpha_3}{\alpha_2} - \theta_3$$

thus it is form of  $a_3 \lambda^3 + a_2 \lambda^2 + a_1 \lambda + a_0 = 0$ . Since  $a_3 = 1$  which is positive by Routh-Hurwitz criteria,  $\lambda$ 's negative if  $a_2 > 0, a_0 > 0, a_2 a_1 - a_0 > 0$ . Each of these conditions are considered next as follows:

(a).  $a_2 > 0, \Rightarrow -(a_{11} + a_{33}) > 0$  this can be satisfied if  $a_{11} < 0$  or  $a_{33} < 0$

i) if  $a_{11} < 0 \Rightarrow \alpha_2 \theta_2 - \alpha_1 \alpha_2 \theta_2 - \alpha_1 \theta_2^2 - \theta_2^2 < 0 \Rightarrow \alpha_2 \theta_2(1 - \alpha_1) - \theta_2^2(\alpha_1 - 1) < 0$  this will hold if  $\alpha_1 < 1$ . In terms of original parameters it represents that  $F < A$ . This implies that, the benefits of predator populations from interaction with prey population is less than natural growth rates of prey population in the absence of predator and scavenger populations.

ii)  $a_{33} < 0 \Rightarrow (\alpha_2^2 \beta_3(\alpha_2 - 1) + (2\beta_1 \alpha_2 \theta_2 - \beta_1 \alpha_2^2)(\theta_3 - \theta_2 \alpha_3) + \alpha_1 \alpha_2 \theta_2(\alpha_3 - \theta_2)(\theta_3 - \beta_2 \alpha_2) + \alpha_2 \theta_2^2(\beta_1 \theta_2 - \beta_3) + \alpha_2 \theta_2 \alpha_3(\alpha_2 - \theta_2) + \theta_2 \theta_3(\alpha_2 - \beta_1 \theta_2)) < 0$

Therefore,  $a_{33} < 0$  if  $\alpha_2 < 1, \theta_3 < \theta_2 \alpha_3, \alpha_3 < \theta_2, \beta_1 \theta_2 < \beta_3, \alpha_2 < \theta_2$  and  $\alpha_2 < \beta_1 \theta_2$

c)  $a_0 > 0 \Rightarrow a_{12} a_{21} a_{33} > 0$  this is satisfied if, for  $a_{33} < 0$  and  $a_{21} > 0$

Clearly,  $a_{12} = -\frac{\beta_1 \theta_2}{\alpha_2}$ , this is negative.

d)  $a_2 a_1 - a_0 > 0 \Rightarrow a_{11} < 0, a_{12} < 0, a_{33} < 0$  and  $a_{11} a_{33} - a_{12} a_{21} < 0$  ( $\nabla$ )

Therefore,  $E_4(\frac{\theta_2}{\alpha_2 - \theta_2}, \frac{\alpha_2((\alpha_2 - \theta_2) - \alpha_1 \theta_2)}{\beta_1(\alpha_2 - \theta_2)^2}, 0)$  is locally asymptotically stable if conditions satisfied.

vi) **Local stability at  $E_5$**

$$J(E_5) = \begin{bmatrix} \frac{(1+x^*)^2 - 2\alpha_1 x^*(1+x^*)^2 - \theta_1 z^*}{(1+x^*)^2} & -\frac{\beta_1 x^*}{1+x^*} & -\frac{\theta_1 x^*}{1+x^*} \\ 0 & \frac{\alpha_2 x^*(1+z^*) + \beta_2 z^*(1+x^*) - \theta_2(1+x^*)(1+z^*)}{(1+x^*)(1+z^*)} & \frac{\beta_3 z^*}{(1+z^*)^2} \\ \frac{\alpha_3 z^*}{(1+x^*)^2} & \beta_3 z^* & \frac{\alpha_3 z^* - 2\theta_4 z^*(1+x^*) - \theta_3(1+x^*)}{1+x^*} \end{bmatrix}$$

The Eigen values of  $J(E_5)$  are obtained by solving

$$\det \begin{bmatrix} a_{11} - \lambda & -a_{12} & -a_{13} \\ 0 & a_{22} - \lambda & -a_{23} \\ -a_{31} & -a_{32} & a_{33} - \lambda \end{bmatrix} = 0$$

Where;  $a_{11} = \frac{(1+x^*)^2 - 2\alpha_1 x^*(1+x^*)^2 - \theta_1 z^*}{(1+x^*)^2}$ ,  $a_{12} = \frac{\beta_1 x^*}{1+x^*}$ ,  $a_{13} = \frac{\theta_1 x^*}{1+x^*}$ ,  $a_{22} = \frac{\alpha_2 x^*(1+z^*) + \beta_2 z^*(1+x^*) - \theta_2(1+x^*)(1+z^*)}{(1+x^*)(1+z^*)}$ ,  $a_{31} = \frac{\alpha_3 z^*}{(1+x^*)^2}$ ,  $a_{32} = \beta_3 z^*$ ,  $a_{33} = \frac{\alpha_3 z^* - 2\theta_4 z^*(1+x^*) - \theta_3(1+x^*)}{1+x^*}$

The characteristic equation becomes

$$\lambda^3 - (a_{11} + a_{22})\lambda^2 + (a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} - a_{13}a_{31})\lambda + a_{12}a_{23}a_{31} + a_{22}a_{13}a_{31} - a_{11}a_{22}a_{33} = 0$$

This is form  $a_3 \lambda^3 + a_2 \lambda^2 + a_1 \lambda + a_0 = 0$

By Routh-Hurwitz criteria the  $\lambda$ 's are negative if  $a_2 > 0, a_0 > 0, a_2 a_1 - a_0 > 0$ . And we considered each of these conditions as follows:

- i)  $a_2 > 0 \Rightarrow -(a_{11} + a_{22} + a_{33}) > 0$  or  $a_{11} + a_{22} + a_{33} < 0$
- ii)  $a_0 > 0 \Rightarrow a_{12}a_{23}a_{31} + a_{22}a_{13}a_{31} > a_{11}a_{22}a_{33}$
- iii)  $a_2 a_1 - a_0 > 0 \Rightarrow -(a_{11} + a_{22})(a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} - a_{13}a_{31}) + a_{11}a_{22}a_{33} - a_{12}a_{23}a_{31} - a_{22}a_{13}a_{31} > 0$  ( $\nabla\nabla$ )

Therefore,  $E_5$  is locally asymptotically stable if conditions (i), (ii) and (iii) holds true.

vii) **Local stability at  $E_7$**

Similar to  $E_5$  after solving Jacobian matrix of system (10) at equilibrium point  $E_7(x^*, y^*, z^*)$ , the eigenvalues are found by solving

$$\det \begin{bmatrix} a_{11} - \lambda & -a_{12} & -a_{13} \\ a_{21} & a_{22} - \lambda & -a_{23} \\ -a_{31} & -a_{32} & a_{33} - \lambda \end{bmatrix} = 0$$

Where;  $a_{11} = \frac{(1+x^*)^2 - 2\alpha_1 x^*(1+x^*)^2 - \theta_1 z^*}{(1+x^*)^2}$ ,  $a_{12} = \frac{\beta_1 x^*}{1+x^*}$ ,  $a_{13} = \frac{\theta_1 x^*}{1+x^*}$ ,  $a_{21} = \frac{\alpha_2 y^*}{(1+x^*)^2}$ ,  $a_{22} = \frac{\alpha_2 x^*(1+z^*) + \beta_2 z^*(1+x^*) - \theta_2(1+x^*)(1+z^*)}{(1+x^*)(1+z^*)}$ ,  $a_{31} = \frac{\alpha_3 z^*}{(1+x^*)^2}$ ,  $a_{32} = \beta_3 z^*$ ,  $a_{33} = \frac{\alpha_3 z^* - 2\theta_4 z^*(1+x^*) - \theta_3(1+x^*)}{1+x^*}$

The characteristic equation becomes

$$\lambda^3 - (a_{11} + a_{22})\lambda^2 + (a_{11} a_{22} + a_{11} a_{33} + a_{22} a_{33} - a_{13} a_{31})\lambda + a_{12} a_{23} a_{31} + a_{22} a_{13} a_{31} - a_{11} a_{22} a_{33} = 0$$

This is form  $a_3 \lambda^3 + a_2 \lambda^2 + a_1 \lambda^1 + a_0 = 0$

By Routh-Hurwitz criteria the  $\lambda$ 's are negative if  $a_2 > 0, a_0 > 0, a_2 a_1 - a_0 > 0$ . And we considered each of these conditions as follows:

- i)  $a_2 > 0 \Rightarrow -(a_{11} + a_{22} + a_{33}) > 0$  or  $a_{11} + a_{22} + a_{33} < 0$
- ii)  $a_0 > 0 \Rightarrow a_{12} a_{23} a_{31} + a_{22} a_{13} a_{31} > a_{11} a_{22} a_{33}$
- iii)  $a_2 a_1 - a_0 > 0 \Rightarrow -(a_{11} + a_{22})(a_{11} a_{22} + a_{11} a_{33} + a_{22} a_{33} - a_{13} a_{31}) + a_{11} a_{22} a_{33} - a_{12} a_{23} a_{31} - a_{22} a_{13} a_{31} > 0$

Therefore,  $E_7$  is locally asymptotically stable if conditions (i), (ii) and (iii) holds true.

### Global Stability Analysis of the Equilibrium Points

**Theorem:** the equilibrium point  $E_4(x^*, y^*, 0)$  is globally asymptotically stable.

**Proof :** Let us consider the following Liapunovs function

$$V(x, y, z) = x - x^* - x^* \ln x/x^* + y - y^* - y^* \ln y/y^* + \frac{z^2}{2}$$

Now the time derivative of V, along the solution of (8/9) can be written as

$$\begin{aligned} \frac{dV}{dt} &= \frac{x - x^*}{x} \frac{dx}{dt} + \frac{y - y^*}{y} \frac{dy}{dt} + z \frac{dz}{dt} \\ &= (x - x^*) \left( 1 - \alpha x - \frac{\beta_1 y}{1+x} \right) + (y - y^*) \left( \frac{\alpha_2 x}{1+x} - \theta_2 \right) \end{aligned}$$

This is also simplified to

$$\frac{dV}{dt} = -\alpha_1(x - x^*)^2 - (\beta_1 - \alpha_2) \frac{(x-x^*)(y-y^*)}{(1+x)(1+x^*)} - \beta_1 \frac{(x-x^*)(xy-x^*y^*)}{(1+x)(1+x^*)}$$

which is negative definite when  $\beta_1 > \alpha_2$

Therefore, with these conditions,  $E_4(x^*, y^*, 0)$  is globally asymptotically stable.

#### 4. Numerical Simulation

##### 4.1 Dynamics of prey and predator populations

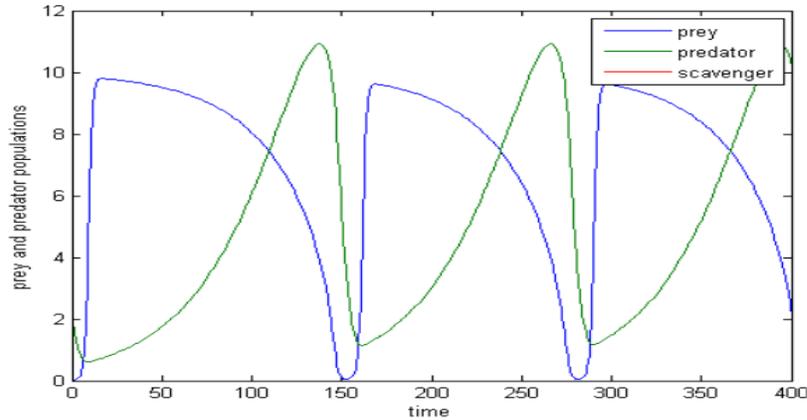
If the scavenger population dies out, the remaining model reverts back to the classical prey predator model. The only difference is functional response included between them.

The model which describes their dynamics are

$$\frac{dx}{dt} = x(1 - \alpha_1 x) - \frac{\beta_1 xy}{(1+x)} \text{----- (11)}$$

$$\frac{dy}{dt} = \frac{\alpha_2 xy}{(1+x)} - \theta_2 y$$

By using  $\alpha_1 = 0.1, \alpha_2 = 0.25, \beta_1 = 0.3, \theta_2 = 0.2$  we can plot the above system of equation (11) as follows



**Figure1: Dynamics of prey and predator without scavenger populations**

In the initial due to less number of prey species the predator species fails downward and prey species dominated for a low rate. The prey abundances are not strongly positively correlated then as one prey species becomes scarce, the predator can continue to feed and increase its population size. As we observe from the above figure when the predator population decreases the prey population conversely increased and vice versa.

#### 4.2 Dynamics of predator and scavenger populations

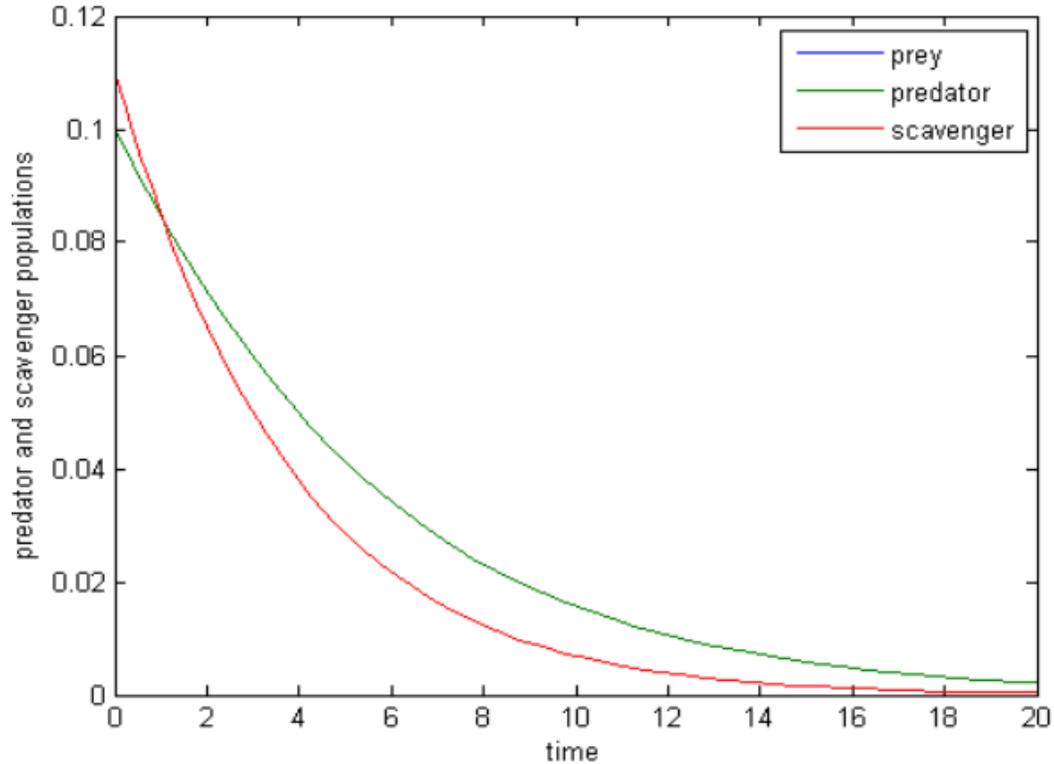
In the absence of prey population the system of differential equation which describes the dynamics between predator and scavenger populations is given by

$$\frac{dy}{dt} = \frac{\beta_2 y z}{(1 + z)} - \theta_2 y \text{ --- (12)}$$

$$\frac{dz}{dt} = \frac{\beta_3 x z}{(1 + y)} - \theta_3 z - \theta_4 z^2$$

One thing that differentiates this paper from other is the benefits of predator species from scavenger species are included. Now we see their dynamics without prey species by using

$$\beta_2 = 0.25 \quad \beta_3 = 0.6 \quad \theta_2 = 0.2 \quad \theta_3 = 0.3 \quad \text{and} \quad \theta_4 = 0,1$$



**Figure 2: Dynamics of predator and scavenger with out prey**

As we observe from the graph without prey population predator as well as scavenger population can exist only for a few times. The scavenger benefit more from interaction between prey and predator, but here the only its source is natural death rates of predator. The predator population benefits from the interaction with scavenger population but not enough for them. In the case of initial both species almost the same and the scavenger species dominates the predator species. But, after a low rates the predator species dominates the scavenger species always, even-though their population decreases from time to time. Furthermore after a long period of time both species almost goes to extinct. From this we conclude that predator and scavenger populations cannot live without prey population for long time.

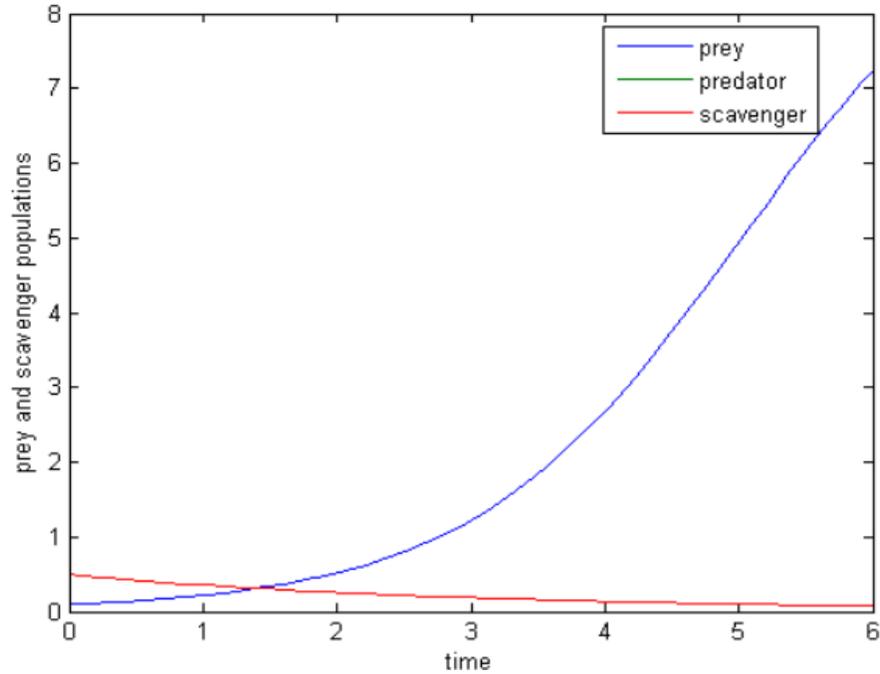
### 4.3 Dynamics of Prey and Scavengers

Similarly, the non-linear differential equation system which shows the dynamics between prey and scavenger populations without predator population is represented as follows,

$$\frac{dx}{dt} = x(1 - \alpha_1 x) - \frac{\theta_1 xz}{(1+x)}$$

$$\frac{dz}{dt} = \frac{\alpha_3 xz}{(1+x)} - \theta_3 z - \theta_4 z^2$$

Using parameters,  $\alpha_1 = 0.1, \alpha_3 = 0.01, \theta_1 = 0.1, \theta_3 = 0.3$ , and  $\theta_4 = 0.1$  to show dynamics of two species:



**Figure 3: Dynamics of prey and scavenger without predator**

#### 4.5. Coexistence of three species

One of the most important question in mathematical biology is concerns the long term survival or coexistence of all the species in multi-species community.

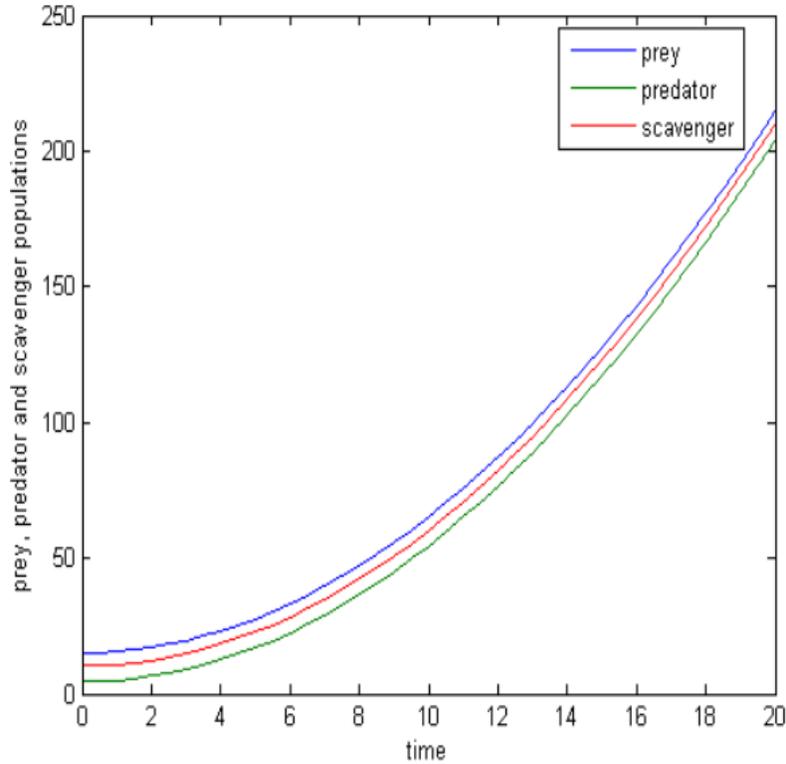
$$\frac{dx}{dt} = x(1 - \alpha_1 x) - \frac{\beta_1 xy}{(1+x)} - \frac{\theta_1 xz}{(1+x)}$$

$$\frac{dy}{dt} = \frac{\alpha_2 xy}{(1+x)} + \frac{\beta_2 xy}{(1+z)} - \theta_2 y$$

$$\frac{dz}{dt} = \frac{\alpha_3 xz}{(1+x)} + \frac{\beta_3 xz}{(1+y)} - \theta_3 z - \theta_4 z^2$$

So, to show coexistence of prey, predator and scavenger population we shall use parameters

$\alpha_1 = 0.1, \alpha_2 = 0.25, \alpha_3 = 0.01, \beta_1 = 0.3, \beta_2 = 0.35, \beta_3 = 0.003, \theta_1 = 0.04, \theta_2 = 0.21, \theta_3 = 0.035, \theta_4 = 0.1$  and initial point  $(x_0, y_0, z_0) = (15.2, 4.6, 10.4)$



**Figure 4: coexistence of three species**

The mechanism that prey, predator and scavenger species coexist is if low predation occurs in an ecosystem. Furthermore, three species become coexist if large number of prey population computes with less number of predator and average number of scavenger populations. As their figure shows all three species increases constantly. Even-though, both predator and scavenger depends on prey species due to large number of prey species their number rise up constantly. Additionally, the main things that maintain their coexistence is both predator and scavenger species cannot get all of the prey species. The scavenger species benefits from the large number prey species as well as from their dead bodies results to increase constantly from time to time. The predator species also gets enough food from other two and their population flows the prey and scavenger species.

### EQUILBRIUM POINTS

In previous section we solved four equilibrium point analysis. Now for these and left two were determine the equilibrium points numerically. We start by setting value for parameter  $\alpha_1 = 0.1$ ,  $\alpha_2 = 0.25$ ,  $\alpha_3 = 0.01$ ,  $\beta_1 = 0.3$ ,  $\beta_2 = 0.4$ ,  $\beta_3 = 0.6$ ,  $\theta_1 = 0.5$ ,  $\theta_2 = 0.2$ ,  $\theta_3 = 0.3$ ,  $\theta_4 = 0.01$ .

Then the equilibriums become,

- i) Trivial equilibrium point(0,0,0)
- ii)  $E_1 = \left(\frac{1}{\alpha}, 0, 0\right) = (10, 0, 0)$
- iii)  $E_2 = \left(\frac{\theta_2}{\alpha_2 - \theta_2}, \frac{\alpha_2((\alpha_2 - \theta_2) - \alpha_1 \theta_2)}{\beta_1(\alpha_2 - \theta_2)^2}, 0\right) = (4, 10, 0)$

iv)  $E_3 = \left(0, \frac{\theta(\beta-\theta)+\theta\theta}{(\beta_3-\theta_3)(\beta_2-\theta_2)-\theta_2\theta_4}, \frac{\theta_2}{\beta_2-\theta_2}\right) = (0,2,1)$

v)  $E_4 = (x^*, 0, z^*)$

In previous section we made difficulty to solve this equilibrium point analytically. Now easily by numerical  $E_4 = (x^*, 0, z^*) = (1.9854,0,7.2185)$

vi) Positivity of  $E_5 = (x^*, y^*, z^*)$  , similar to  $E_4$  it is difficult to solve analytically, but by using MATLAB computer program

$$E_5 = (x^*, y^*, z^*) = (0.0321, 1.8874, 0.9251)$$

Equilibrium	Existence condition	Stability condition
$E_0$	Always exist	saddle
$E_1$	Always exist	If $\theta_2 < \frac{\alpha_2}{1+\alpha_1}$ and $\theta_3 < \frac{\alpha_3}{1+\alpha_1}$
$E_2$	If $\theta_2 < \alpha_2$ and $\alpha_1\theta_2 < \alpha_2 - \theta_2$	if(∇) holds
$E_3$	If $\theta_2 < \beta_2, \theta_3 < \beta_3$ and $\theta_2\theta_4 < (\beta_3 - \theta_3)(\beta_2 - \theta_2)$	if (∇∇)holds

### Result and discussion

In this section the stability of the equilibrium points were discussed and the result of the study was described in both analytically and numerically. The following condition is needed to analysis the equilibrium points of the system. The results in our study coincide with **Nolting B.et al.,(2008)**.

$p(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$  be characteristic equation for matrix defined in (10). Then the following statements are true.

- i) If every roots of the characteristic equation is less than one, then the equilibrium point of the system (9) is locally asymptotically stable and equilibrium is called sink
- ii) If at least one of the root has absolute value greater than one, then the equilibrium point of system (9) is unstable and the equilibrium is called saddle.
- iii) If every root of system has absolute value greater than one, then the system is unstable and the equilibrium is called source.

Using this lemma we check the stability analysis numerically at give equilibrium points;

### Local stability of $E_0$

we shall use parameters  $\alpha_1 = 0.1, \alpha_2 = 0.25, \alpha_3 = 0.01, \beta_1 = 0.3, \beta_2 = 0.4, \beta_3 = 0.06, \theta_1 = 0.5, \theta_2 = 0.2, \theta_3 = 0.3, \theta_4 = 0.01$ . and  $E_0(0.2,0.1,0.5)$  and substituting in the Jacobean matrix (10) this set of parameter values the Eigen value becomes  $\lambda_1 = 1, \lambda_2 = -0.2, \lambda_3 = -0.3$ . This implies the equilibrium  $E_0(0.2,0.1,0.5)$  is unstable. Now by using above parameters we show their graph as follows.

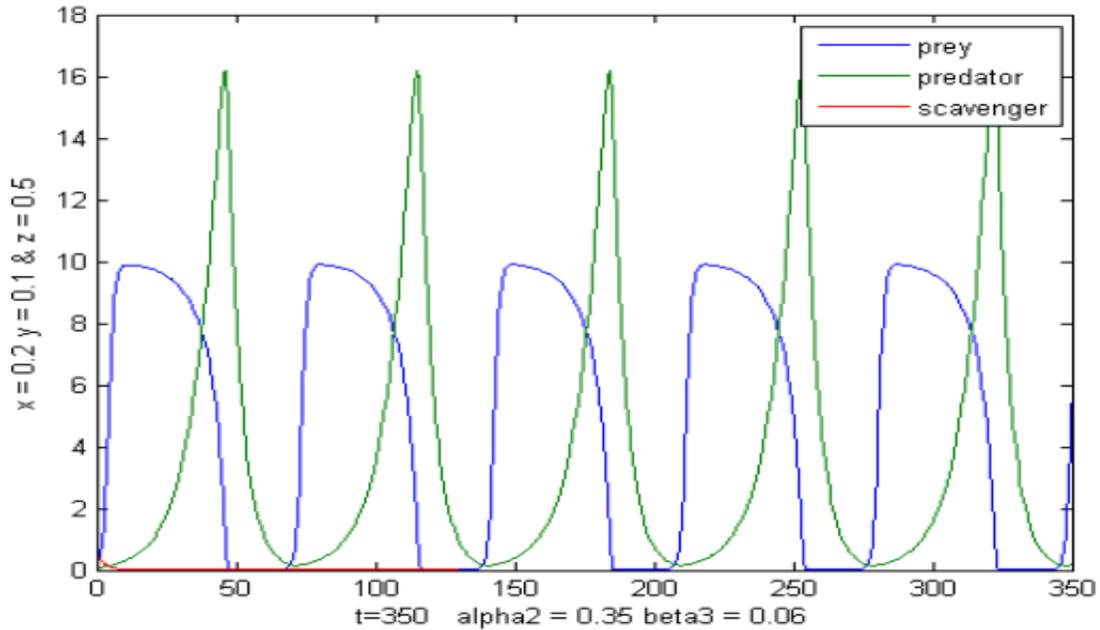


Figure 5: Time series plot for unstable equilibrium point  $E_0$

### local stability of $E_1$

Local stability of  $E_1$  we shall use  $\beta_3 = 0.06, \theta_2 = 0.25, \theta_3 = 0.3$  and other as before we get eigen value  $\lambda_1 = -1, \lambda_2 = -0.023, \lambda_3 = -0.29$  and  $E_1(9.7, 0.5, 0.6)$ . hence with this parameter  $E_1$  is asymptotically stable and their graph is shown as

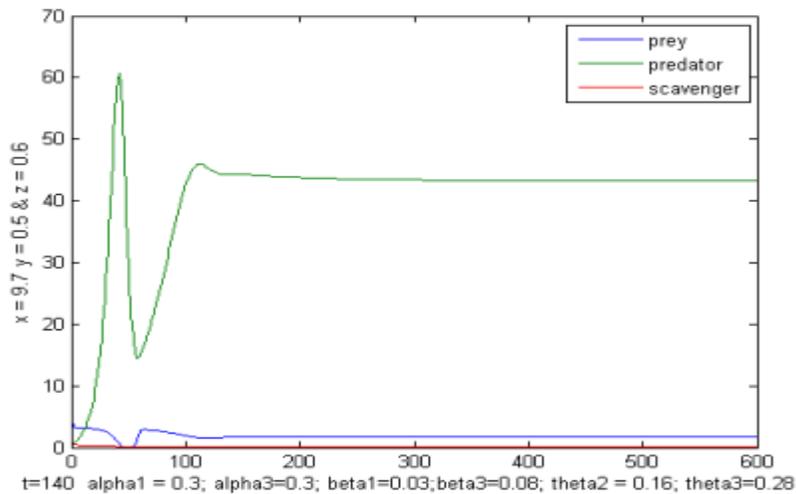
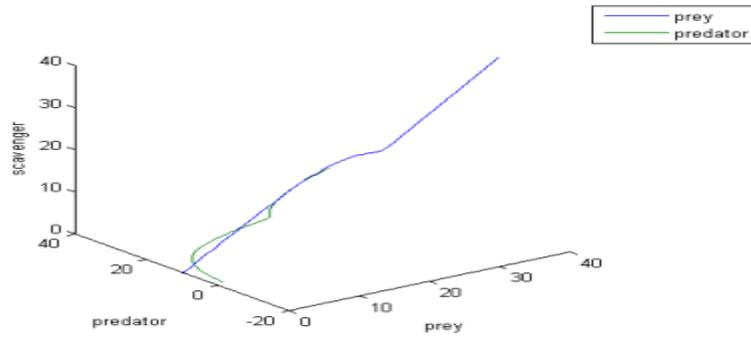
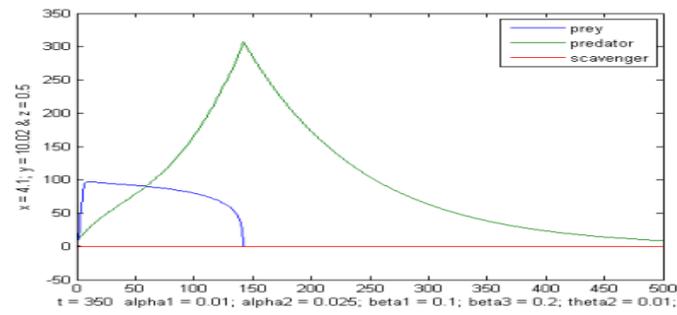


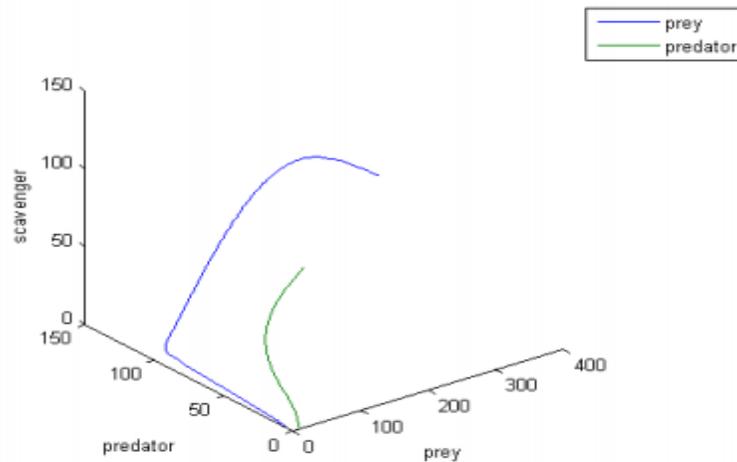
Figure 6: time series plot for stable equilibrium point



**Figure 7: Phase portrait at equilibrium point E\_1**



**Figure 8: Time series plot for stability at equilibrium point E\_2**



**Figure 9: Phase portrait at equilibrium E\_2**

Mathematical Modeling of the dynamics of Prey-Predator with Scavenger in a closed habitat

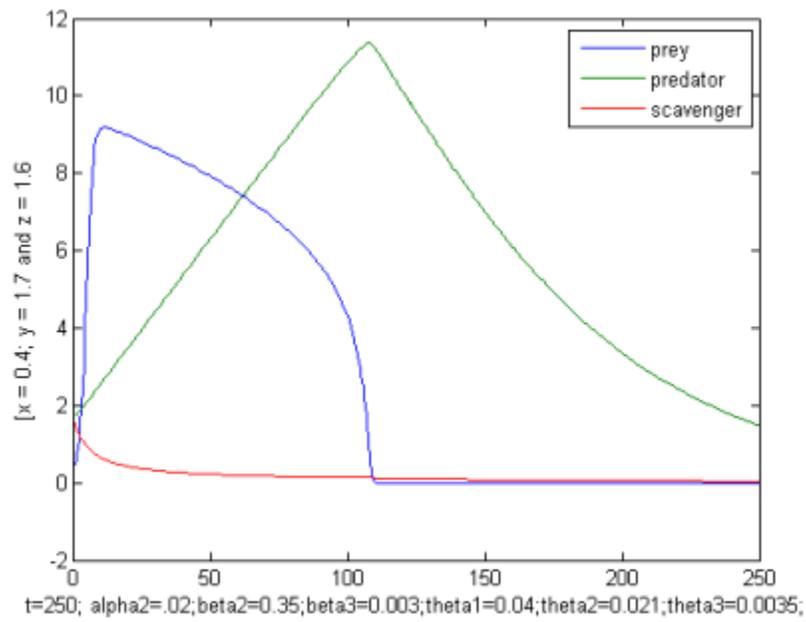


Figure 10: Time series plot for stability equilibrium point E\_3

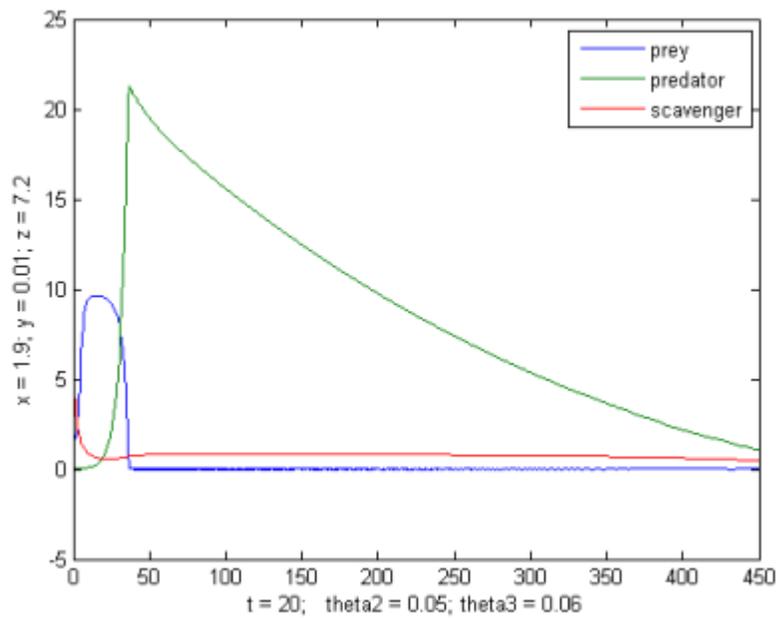
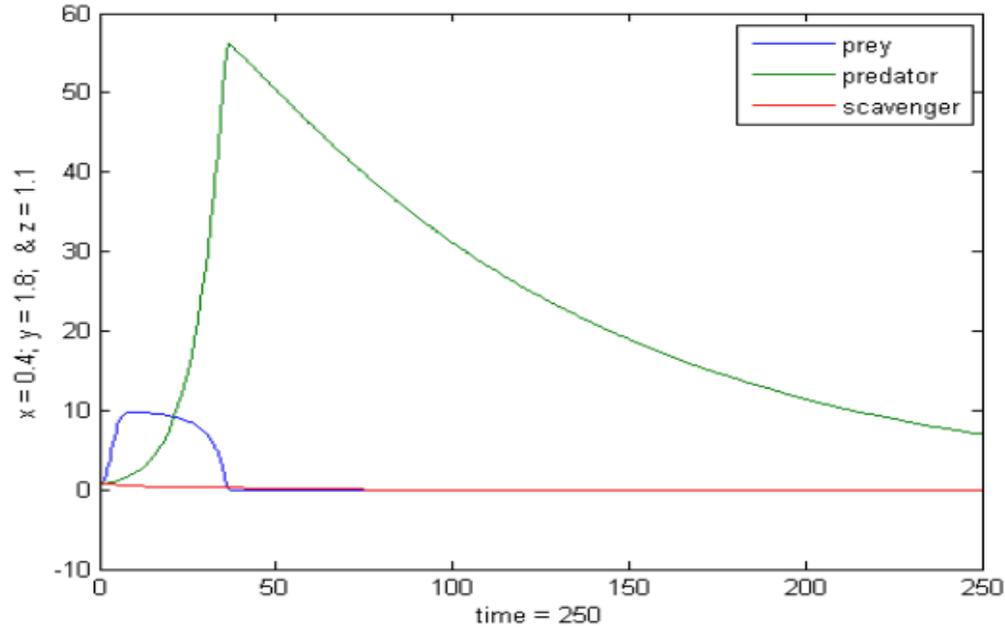


Figure 11: Time series plot for stability equilibrium point E\_4



**Figure 12:Time serious plot for stability equilibrium point E\_5**

## CONCLUSION

Modified Lotka Voltera model of three species is discussed in this paper. These three species are prey, predator and scavenger in some closed ecology. In this model these three species interact in ecology non-linearly governed by ordinary differential equation. In this paper we discussed, three species are coexisting in some conditions. Here we found that predator and prey can coexist in absence of scavenger. In contrary, scavenger and predator cannot exist without prey. This is because prey is food for both the rest species. Finally, the main aim of this paper is to analysis the coexistence of these three species. Thus, these species coexist in two main ways: they will cycle in between two stable population or the populations will cycle until they get saturated and remain constant.

## REFERENCE

1. **Chauvet, E. ,Pullet, J.E., Previte, J.P, Walls Z., (2002, October).** A Lotka-Volterra Three-species Food Chain Mathematics Magazine, 75(4), 243-255. Retrieved from <http://math.bd.psu.edu/faculty/jprevite/mathmag243-255.pdf>.
2. **BoyceW.E.&DiPrimaR.C.,(1977).**Elementary Differential Equations and Boundary Value Problems.(3rd ed). New York NY: John Wiley & Sons,Inc.
3. **R. Arditi, L.R. Ginzburg,** Coupling in predator-prey dynamics: ratio-dependence, J.Theor. Biol (1989).
4. **Chauvet, E. ,Pullet, J.E., Previte, J.P, Walls Z.,** Journal of Mathematical Analysis and **T. L. Devault, O. E. Rhodes, and J. A. Shivik.** Scavenging by vertebrates: Behavioral,

- ecological, and evolutionary perspectives on an important energy transfer pathway in terrestrial systems (2011) Applications,(2002, October)
5. **B. Dubey, B. Das, J. Hussain**, A predator-prey interaction model with self and cross diffusion, *Ecol. Model* (2001).
  6. **M. Fan, Y. Kuang**, Dynamics of a non-autonomous predator-prey system with Beddington DeAngelis functional response, *Journal of Mathematical Analysis and applications*
  7. **K. Fujii**. Complexity-stability relationship of two-prey{one-predator species system model: local and global stability. *J. Theor. Biol* (2005). (2004).
  8. **G. W. Harrison**. Global stability of food chains. *Am. Nat* (2012).
  9. **Murray, J.D.** *Mathematical Biology: I. an Introduction* (2000).
  10. **Korobeinikov,A.,Wake.,G.** Global properties of the three-dimensional predator-prey
  11. Lotka-Volterra systems. *Journal of Applied Mathematics and Decision Sciences* (1999).
  12. **N. Krikorian**. The Volterra model for three species predator-prey systems: boundedness
  13. and stability. *J. Math. Biol* (2004).
  14. **Nolting, B., Paullet, J., Previte, J. (2008)**. Introducing a scavenger onto predator prey model. *Applied Math E-Notes* 8, 214-222 Retrieved from <http://math.bd.psu.edu>
  15. **LaSalle, J. P. and S. Lefschetz**. *Stability by Liapunov's Direct Method with Applications*. New York: Academic Press (2001).
  16. **Y. Morozov** and B. L. Li. Parametric analysis of a predator-prey system stabilized
  17. by a top predator. *J. Math. Biol*, 2006.
  18. **Nolting, B., Paullet, J., Previte, J.** Introducing a scavenger onto predator prey
  19. model. *Applied Math E-Notes* (2008).
  20. **Previte, J.P, Homan K.A.,** Chaos in a Predator-Prey Model with an Omnivore(2010).
  21. **Previte, J.P, Homan K.A.** Period Doubling Cascades in a Predator-Prey Model with
  22. a Scavenger. *SIAM Review* (2013).
  23. **S. Xing and X. Zhongyi**. The prey-dependent consumption two-prey one-predator models with stage structure for the predator and impulsive effects. *J. of Theor. Biology*, 2006.