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Research Article

Numerical Approximation of the Average Run Length on a Modified EWMA Control Chart for an ARX(1,1) Process with Exponential White Noise

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Abstract

The purpose of this study is to evaluate the average run length (ARL) by using a numerical integral equation (NIE) method based on the Gauss-Legendre quadrature rule on a modified exponentially weighted moving average (EWMA) control chart for a first-order autoregressive process with an explanatory variable (ARX(1,1)) with exponential white noise. In addition, the performance of the modified EWMA control chart for this process was compared with the standard EWMA control chart based on the relative mean index (RMI), in which the modified EWMA control chart was found to be better than the standard EWMA control chart for all smoothing parameter values.

Keywords: *Autoregressive process, Explanatory variable, Modified EWMA control chart, Numerical integral equation*

Introduction

Statistical process control (SPC) is the application of statistical methods to monitor, measure, control, and improve the quality of a process to ensure that it operates at its full potential [1]. It plays an essential role in all aspects of the manufacturing industry as well as various other fields. The control chart is an SPC tool used for monitoring processes and detecting shifts in the process mean. Shewhart [2] introduced the Shewhart control chart, which is widely used for detecting large shifts in the process mean. However, for small changes in the process mean, suitable control charts are the cumulative sum (CUSUM) [3] and the exponentially weighted moving average (EWMA) control charts [4]. Moreover, the benefits of using the EWMA control chart have been widely reported [5]–[9].

Recently, Khan et al. [10] developed a new modified EWMA control chart based on the modified EWMA statistic [11], for which they considered the past and current behavior of the process. They compared it with the standard and modified [11] EWMA control charts and found that their proposed control chart was more efficient in terms of the average run length (*ARL*) (a popular measure for control chart performance) and can detect shifts more quickly.

The *ARL* is the average number of observations taken from an in-control process until the control chart signals that it is

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out-of-control. The *ARL* is categorized as in-control (ARL_0) or out-of-control (ARL_1). ARL_0 is the average number of observations when the process is in-control and should be as large as possible. ARL_1 is the average number of observations when the process is out-of-control and should be as small as possible. Various methods to estimate the *ARL* have been reported, such as Monte Carlo simulation, Markov chains, Martingales, and numerical integration equations (NIEs) based on several quadrature rules, namely midpoint, trapezoidal, Simson's rule, and Gauss-Legendre [12].
Several researchers have focused on approximations of the *ARL* to measure the efficacy of control charts by using many methods. Robert [4] used Monte Carlo simulations to estimate the *ARL* for a EWMA control chart. Crowder [5] used an NIE approach to measure the *ARL* for a Gaussian distribution. Harris and Ross [13] studied the *ARL* for a CUSUM control chart for a process with serially correlated observations via Monte Carlo simulations. Montgomery and Mastrangelo [14] evaluated the performance of EWMA control charts for serially correlated processes by determining the *ARL* using Monte Carlo simulations. Sukparungsee and Novikov [15] studied the detection of changes in a process parameter on a EWMA control chart by using a Martingale approach to derive approximate analytical formulas for the *ARL* and the average delay. Mititelu et al. [12] used a linear Fredholm-type integral equation approach to derive explicit formulas for the *ARLs* on a CUSUM control chart when random observations follow a hyperexponential distribution and on a EWMA control chart with observations following a Laplace distribution. Paichit [16] used an NIE to find the exact expression for the *ARL* of an EWMA control chart for an autoregressive process with exogenous input (ARX(p)). Phanthuna et al. [17] derived explicit formulas for the *ARL* of a modified EWMA control chart for an exponential AR(1) process. Explicit formulas of numerical approximations for the *ARL* of modified EWMA control chart for a first-order moving average (MA(1)) process were presented by Supharakonsakun et al. [18].
The objective of this study is to propose an NIE method for the *ARL* on a modified EWMA control chart for a first-order autoregressive process with an explanatory variable (ARX(1,1)) with exponential white noise via the Gauss-Legendre quadrature rule. In addition, we compared the performances of the modified and standard EWMA control charts based on the relative mean index (RMI).

EWMA Control Charts for an ARX(1,1) Process with Exponential White Noise

A. The EWMA Control Chart

The EWMA control chart is often used for monitoring and detecting small shifts in the process mean [4]. It can be expressed with a recursive equation as follows:

$$Z_t = (1-\lambda)Z_{t-1} + \lambda Y_t \qquad ;t = 1,2,3,..., \tag{1}$$

where Z_t is the EWMA statistic, Y_t is the sequence of the ARX(1,1) process with exponential white noise, and λ is an exponential smoothing parameter $(0 < \lambda \le 1)$. The expected value and the variance of the EWMA control chart are $E(Z_t) = \mu_0$ and $Var(Z_t) = \left(\frac{\lambda}{2-\lambda}\right)\sigma^2$, respectively. Therefore, the upper control limit (UCL) and the lower control limit (LCL) of the EWMA control chart are

$$LCL = \mu_0 - L\sigma\sqrt{\frac{\lambda}{(2-\lambda)}} \tag{2a}$$

and $UCL = \mu_0 + L\sigma\sqrt{\frac{\lambda}{(2-\lambda)}}$, (2b) respectively,

where L is an appropriate control width limit $(L > 0)$.

The stopping time τ_b for the EWMA control chart can be written as

$$\tau_b = \inf\{t > 0;\ Z_t > b\}, \quad (3)$$

where b is a constant known as the UCLfor an ARX(1,1) process *ARL*. The upper side of the $(b > 0)$ on a EWMA control chart with an initial value $(Z_0 = u)$ can be found. Now, we define the function as $L(u)$

$$L(u) = ARL = E_\infty(\tau_b). \quad (4)$$

B. The Modified EWMA Control Chart

The modified EWMA control chart [10] based on the modified EWMA statistic [11] is a simplified EWMA control chart for detecting shifts in the process mean under the assumption that the observations follow a normal distribution. The modified EWMA control chart [10] is defined as

$M_t = (1-\lambda)M_{t-1} + \lambda Y_t + k(Y_t - Y_{t-1}) \quad ; t = 1,2,3,...$, (5) where M_t is the modified EWMA statistic, Y_t is the sequence of the ARX(1,1) process with exponential white noise, λ is an exponential smoothing parameter $(0 < \lambda \le 1)$, and k is a constant $(k > 0)$. The expected value and the variance of the modified EWMA control chart are $E(M_t) = \mu_0$ and $Var(M_t) = \frac{(\lambda + 2\lambda k + 2k^2)\sigma^2}{(2-\lambda)}$, respectively. Therefore, the LCL and UCL of the modified EWMA control chart can be written as

$$LCL = \mu_0 - L\sigma\sqrt{\frac{(\lambda + 2\lambda k + 2k^2)}{(2-\lambda)}} \quad (6a)$$

and $UCL = \mu_0 + L\sigma\sqrt{\frac{(\lambda + 2\lambda k + 2k^2)}{(2-\lambda)}}$, (6b)

respectively, where L is an appropriate control width limit $(L > 0)$.

The stopping time τ_h for the modified EWMA control chart can be written as

$$\tau_h = \inf\{t > 0;\ M_t > h\}, \quad (7)$$

where h is a constant known as the UCL $(h > 0)$. The upper side of the ARL for an ARX(1,1) process on a modified EWMA control chart with an initial value ($M_0 = u$) can be found. Now, the function $L(u)$ can be defined as

$$ARL = G(u) = E_\infty(\tau_h). \quad (8)$$

C. An ARX(1,1) Process with Exponential White Noise

This can be written as

$$Y_t = \delta + \phi Y_{t-1} + \beta X + \varepsilon_t \quad ; t = 1,2,3,..., \quad (9)$$

where δ is a constant $(\delta \ge 0)$, ϕ is an autoregressive coefficient $(-1 < \phi < 1)$, ε_t is a white noise process ($\varepsilon_t \sim Exp(\alpha)$), X are explanatory variable of Y_t, and β are coefficients of X.

Approximating the ARL on a Modified EWMA Control Chart

A. The ARL on a EWMA Control Chart

This can be derived by using a Fredholm integral equation of the second kind [12], and thus the formula for $L(u)$ can be written as

$$L(u)=1+\frac{1}{\lambda}\int_0^b L(w)\, f\left\{\begin{matrix}\dfrac{w-(1-\lambda)u}{\lambda}\\ -\delta-\phi Y_{t-1}-\beta X\end{matrix}\right\}dw \qquad (10)$$

We use the quadrature rule approach for the numerical method to solve the integral equation, which can be approximated using the midpoint rule. The approximation for the integral is in the form

$$\int_0^b L(w)\, f(w)\,dw \approx \sum_{j=1}^{m} w_j f(a_j) \qquad (11)$$

Thus, integral equation $L(u)$ can be approximated as

$$\tilde{L}(a_i)=1+\frac{1}{\lambda}\sum_{j=1}^{m} w_j \tilde{L}(a_j)\, f\left\{\begin{matrix}\dfrac{a_j-(1-\lambda)a_i}{\lambda}\\ -\delta-\phi Y_{t-1}-\beta X\end{matrix}\right\};\ i=1,2,...,m. \qquad (12)$$

We can write the numerical approximation for the integral equation in matrix form as $\mathbf{L}_{m\times 1}=\left(\mathbf{I}_m-\mathbf{R}_{m\times m}\right)^{-1}\mathbf{1}_{m\times 1}$. Hence, an approximation of the NIE method for the ARX(1,1) process on a EWMA control chart can be derived as

$$\tilde{L}(u)=1+\frac{1}{\lambda}\sum_{j=1}^{m} w_j \tilde{L}(a_j)\, f\left\{\begin{matrix}\dfrac{a_j-(1-\lambda)u}{\lambda}\\ -\delta-\phi Y_{t-1}-\beta X\end{matrix}\right\}, \qquad (13)$$

where $a_j=\frac{b}{m}\left(j-\frac{1}{2}\right)$ and $w_j=\frac{b}{m};\ j=1,2,...,m$.

B. The ARL of a process on a modified EWMA Control Chart

This can be derived by using a Fredholm integral equation of the second kind [12], and so the formula for $G(u)$ can be written as

$$G(u)=1+\frac{1}{\lambda+k}\int_0^h G(w)\, f\left\{\begin{matrix}\dfrac{w-(1-\lambda)u}{(\lambda+k)}+\dfrac{kY_{t-1}}{(\lambda+k)}\\ -\delta-\phi Y_{t-1}-\beta X\end{matrix}\right\}dw. \qquad (14)$$

We use the quadrature rule approach for the numerical method to solve the integral, which can be approximated using the midpoint rule. The approximation of the integral is in the form

$$\int_0^h G(w)\, f(w)\,dw \approx \sum_{j=1}^{m} w_j f(a_j) \qquad (15)$$

Thus, integral equation $G(u)$ can be approximated as

$$\tilde{G}(a_i)=1+\frac{1}{\lambda+k}\sum_{j=1}^{m} w_j \tilde{G}(a_j)\, f\left\{\begin{matrix}\dfrac{a_j-(1-\lambda)a_i}{(\lambda+k)}+\dfrac{kY_{t-1}}{(\lambda+k)}\\ -\delta-\phi Y_{t-1}-\beta X\end{matrix}\right\};\ i=1,2,...,m. \qquad (16)$$

We can write the numerical approximation for the integral equation in matrix form as $\mathbf{G}_{m\times 1}=\left(\mathbf{I}_m-\mathbf{R}_{m\times m}\right)^{-1}\mathbf{1}_{m\times 1}$. Therefore, an approximation of the NIE method for an ARX(1,1) process on a modified EWMA control chart can be written as

$$\tilde{G}(u)=1+\frac{1}{\lambda+k}\sum_{j=1}^{m}w_j\tilde{G}(a_j)\,f\left\{\begin{array}{l}\dfrac{a_j-(1-\lambda)u}{(\lambda+k)}+\dfrac{kY_{t-1}}{(\lambda+k)}\\ -\delta-\phi Y_{t-1}-\beta X\end{array}\right\}, \quad (17)$$

where $a_j=\frac{h}{m}\left(j-\frac{1}{2}\right)$ and $w_j=\frac{h}{m};\ j=1,2,...,m$

Numerical Results

We used simulated data and the RMI to compare the performances of the *ARL* of an ARX(1,1) process with exponential white noise on standard and modified EWMA control charts. The RMI is defined as

$$RMI=\frac{1}{n}\sum_{i=1}^{n}\left(\frac{ARL_{shift,i}-Min\left[ARL_{shift,i}\right]}{Min\left[ARL_{shift,i}\right]}\right), \quad (18)$$

where $ARL_{shift,i}$ is the *ARL* of the control chart when a shift in the process mean is detected; $shift,i$ is the shift size for $i=1,2,...,n,$ and $Min\left[ARL_{shift,i}\right]$ denotes the smallest *ARL* of two control charts in the comparison when the position process shift. The control chart with the smallest RMI performs the best for detecting a process mean change.

Equations (13) and (17) are used to evaluate the *ARL* for an ARX(1,1) process with exponential white noise on standard and modified EWMA control charts, respectively, when the number of divisions (*m*) is 1,000 iterations. The results for the *ARL* on a modified EWMA control chart are reported in Table I, while a performance comparison of the *ARL* on standard and modified EWMA control charts is given in Tables II–V.

Table I. The ARL_0 on the modified EWMA control chart for an ARX(1,1) process for $\delta=2$

λ	ϕ	k	h	ARL_0	CPU Time (second)
		1	0.24685787	370.005720	10.172
	0.2	3	0.74194834	370.006068	10.234
0.05		5	1.23700130	370.006883	10.047
		1	0.36945560	370.008758	10.250
	-0.2	3	1.11038960	370.009474	10.359
		5	1.85128190	370.009121	10.078
		1	0.24794084	370.004985	10.110
	0.2	3	0.74430970	370.002031	10.296
0.075		5	1.24084915	370.005379	10.297
		1	0.37158220	370.006695	9.968
	-0.2	3	1.11558720	370.004886	9.954
		5	1.85986380	370.007597	10.296
		1	0.24911517	370.006842	10.188
	0.2	3	0.74670150	370.003585	10.156
0.1		5	1.24472095	370.005406	10.329
		1	0.37383660	370.006482	10.219
	-0.2	3	1.12083830	370.004791	10.125
		5	1.86851220	370.006463	10.141
		1	0.25460320	370.006132	9.985
	0.2	3	0.75656450	370.003000	10.485
0.2		5	1.26044810	370.003643	9.875
		1	0.38397800	370.002433	10.140

-0.2	3	1.14237590	370.007805	10.187
	5	1.90378480	370.002336	9.797

The results in Table I are for the *ARL₀* evaluated by using Equation (17) on a modified EWMA control chart when $\delta = 2$; $\phi = 0.2$ or -0.2; and $\lambda =$ 0.05, 0.075, 0.10 or 0.20 for $ARL_0 = 370$.

Table II. Comparison of the *ARL* on standard and modified EWMA control charts for an ARX(1,1) process with $\lambda = 0.05$ and $ARL_0 = 370$

Shift Size	EWMA	Modified EWMA		
		k=1	k=3	k=5
0.000	370.006258 (9.921)[a]	370.005720 (10.172)	370.006068 (10.234)	370.006883 (10.047)
0.001	361.840299 (10.172)	262.619518 (9.938)	213.998371 (10.250)	203.039021 (10.719)
0.005	331.087697 (9.985)	121.498788 (9.096)	79.997223 (10.016)	72.790092 (10.156)
0.01	296.592559 (10.094)	72.658141 (10.640)	45.115649 (10.375)	40.677762 (10.859)
0.03	193.088007 (9.891)	27.846438 (9.922)	16.798341 (10.031)	15.118578 (10.188)
0.05	127.846224 (9.496)	17.234389 (10.156)	10.536846 (10.046)	9.526283 (9.750)
0.07	86.042790 (10.015)	12.496648 (9.734)	7.788153 (9.969)	7.077401 (10.094)
0.10	48.944703 (10.078)	8.876678 (9.906)	5.702947 (10.188)	5.221445 (9.719)
0.30	3.033964 (10.032)	3.236575 (9.703)	2.444583 (9.922)	2.318612 (10.000)
0.50	1.196325 (10.344)	2.179442 (9.860)	1.810107 (10.406)	1.749043 (10.328)
RMI	**6.072619**	**0.675917**	**0.137701**	**0.051335**

[a] The computations for the NIE method were carried out on a Windows 10 Professional 64-bit with RAM of 8 GB and an Intel Core i5 CPU.

Table III. Comparison of the *ARL* on standard and modified EWMA control charts for an ARX(1,1) process with $\lambda = 0.075$ and $ARL_0 = 370$

Shift Size	EWMA	Modified EWMA		
		k=1	k=3	k=5
0.000	370.002108 (10.000)	370.004985 (10.110)	370.002031 (10.296)	370.005379 (10.297)
0.001	364.245460 (10.140)	259.017209 (10.406)	213.174045 (9.797)	202.731728 (10.563)
0.005	342.206459 (10.156)	117.737595 (10.140)	79.432743 (10.219)	72.596607 (10.156)
0.01	316.741648 (9.954)	70.006755 (10.469)	44.763817 (10.360)	40.559644 (10.078)
0.03	234.216144 (9.734)	26.723979 (9.812)	16.662916 (10.250)	15.073779 (10.703)
0.05	175.196497 (9.984)	16.543962 (10.297)	10.454507 (10.093)	9.499085 (10.125)
0.07	132.493673 (9.640)	12.007262 (10.391)	7.729683 (10.234)	7.058076 (10.328)
0.10	88.853250	8.543892	5.662781	5.208144

	(10.125)	(10.078)	(10.172)	(10.063)
0.30	10.464511	3.149626	2.433086	2.314745
	(10.078)	(10.265)	(10.125)	(10.578)
0.50	2.809614	2.136856	1.804023	1.746970
	(9.844)	(10.407)	(10.329)	(10.344)
RMI	**9.029192**	**0.562837**	**0.080174**	**0.000000**

Table IV. Comparison of the *ARL* on standard and modified EWMA control charts for an ARX(1,1) process with $\lambda = 0.1$ and $ARL_0 = 370$

Shift Size	EWMA	Modified EWMA		
		k=1	k=3	k=5
0.000	370.007602	370.006842	370.003585	370.005406
	(10.125)	(10.188)	(10.156)	(10.329)
0.001	365.408837	255.519339	212.365677	202.429597
	(9.797)	(10.421)	(9.937)	(10.093)
0.005	347.664589	114.210070	78.881810	72.406592
	(9.921)	(10.048)	(10.125)	(10.109)
0.01	326.871813	67.548996	44.420927	40.443694
	(10.672)	(10.297)	(10.563)	(10.000)
0.03	256.946295	25.694031	16.531085	15.029819
	(9.719)	(9.563)	(10.625)	(10.156)
0.05	203.841181	15.911694	10.374369	9.472398
	(9.984)	(9.750)	(10.109)	(9.875)
0.07	163.123161	11.559372	7.672778	7.039115
	(10.610)	(10.047)	(9.875)	(10.155)
0.10	118.580326	8.239358	5.623691	5.195095
	(10.062)	(10.547)	(10.344)	(10.579)
0.30	20.945594	3.069845	2.421892	2.310951
	(10.156)	(9.828)	(10.109)	(10.421)
0.50	6.336435	2.097677	1.798097	1.744938
	(10.265)	(9.984)	(10.516)	(10.110)
RMI	**11.444254**	**0.517537**	**0.075883**	**0.000000**

The results in Tables II–V show a comparison of the approximate *ARLs* on standard and modified EWMA control charts given $\delta = 2$; $\phi = 0.2$; and shift size = 0.00, 0.001, 0.005, 0.01, 0.03, 0.05, 0.07, 0.10, 0.30 or 0.50 for $ARL_0 = 370$ and $\lambda =$ 0.05, 0.075, 0.10 and 0.20. When $\lambda = 0.05$, the performance of the modified EWMA was better than that of the standard EWMA control chart except for a shift size of 0.5. When $\lambda = 0.75,$ 0.10 or 0.20, the performances of the modified EWMA control chart were better than those of the standard EWMA control chart for all shift sizes and for all k. The RMI values of the modified EWMA control chart were less than those of the EWMA control chart for all k. In addition, as k increased, the ARL_1 and the RMI decreased.

Table V. Comparison of the *ARL* on standard and modified EWMA control charts for an ARX(1,1) process with $\lambda = 0.2$ and $ARL_0 = 370$

Shift Size	EWMA	Modified EWMA		
		k=1	k=3	k=5
0.000	370.005187	370.006132	370.003000	370.003643

	(9.922)	(9.985)	(10.485)	(9.875)
0.001	358.854692	242.538466	209.266310	201.266638
	(10.109)	(9.938)	(10.062)	(10.266)
0.005	319.519457	102.104856	76.803257	71.679578
	(9.813)	(10.640)	(10.344)	(10.078)
0.01	279.736316	59.321760	43.132479	40.000695
	(9.344)	(9.735)	(10.375)	(10.109)
0.03	180.425019	22.318082	16.037244	14.862057
	(9.875)	(10.110)	(10.157)	(10.328)
0.05	127.786741	13.847480	10.074335	9.370582
	(10.141)	(9.579)	(10.484)	(10.594)
0.07	95.746261	10.098792	7.459761	6.966785
	(9.672)	(10.015)	(10.406)	(10.016)
0.10	66.474164	7.246386	5.477362	5.145319
	(9.828)	(10.266)	(10.187)	(10.235)
0.30	14.385492	2.808121	2.379954	2.296486
	(9.923)	(10.203)	(10.062)	(9.843)
0.50	6.020100	1.968405	1.775876	1.737188
	(9.844)	(10.187)	(10.797)	(10.141)
RMI	**7.378114**	**0.367309**	**0.059734**	**0.000000**

Conclusion

An NIE method for the *ARL* on a modified EWMA control chart for an ARX(1,1) process with exponential white noise was presented. Moreover, the presented procedure was considerably closer to the computational time of the traditional procedure. Especially, the NIE method for the *ARL* on a modified EWMA control chart could detect small shifts in the process mean better than on the standard EWMA control chart. When $\lambda = 0.075,$ 0.10, or 0.20 for all shift sizes, the monitoring on the modified EWMA control chart performed better than the EWMA control chart for all constant *k*. Moreover, based on the RMI value, it is evident that the modified EWMA control chart outperformed the standard EWMA control chart for all λ. Furthermore, we discovered that the modified EWMA control chart performance depends on optimal values for constants *k* and λ.

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